# P and NP
# CISC5835, Algorithms for Big Data
# CIS, Fordham Univ.

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# Efficient Algorithms

- So far, we have developed algorithms for finding
  - shortest paths in graphs,
  - minimum spanning trees in graphs,
  - matchings in bipartite graphs,
  - maximum increasing subsequences,
  - maximum flows in networks,
  - …
- All these algorithms are efficient, because in each case their time requirement grows as a polynomial function (such as $n$, $n^2$, or $n^3$) of the size of the input ($n$).
  - These problems are tractable.

# Exponential search space

- In all these problems we are searching for a solution (path, tree, matching, etc.) from among an exponential number of possibilities.
  - Brute force solution: checking through all candidate solutions, one by one.
  - Running time is $2^n$, or worse, useless in practice
- Quest for efficient algorithms: finding clever ways to bypass exhaustive search, using clues from input in order to dramatically narrow down the search space.
  - for many problems, this quest hasn't been successful: fastest algorithms we know for them are all exponential.

# Satisfiability Problem

- A boolean expression in **conjunctive normal form (CNF)**

$$(x \vee y \vee \bar{z})(x \vee \bar{y})(y \vee \bar{z})(z \vee \bar{x})(\bar{x} \vee \bar{y} \vee \bar{z})$$

  - **literals**: a boolean variable or negation of one
  - a collection of clauses (in parentheses), each consisting of **disjunction** (logical or, $\vee$) of several literals
- A satisfying truth assignment: an assignment of false or true to each variable so that every clause contains a literal whose value is true, and whole expression is satisfied (true)
  - is (x=T, y=T, z=F) satisfying truth assignment to above CNF?
- **SAT Problem**
  - Given a Boolean formula in CNF
  - Either find a satisfying truth assignment or report that none exists.

# SAT as a search problem

- SAT is a typical **search problem (or decision problem)**
  - Given an instance I (i.e., some input data specifying problem at hand),
  - To find a solution S (an object that meets a particular specification). If no such solution exists, we must say so.
- **In SAT:** input data is a Boolean formula in conjunctive normal form, and solution we are searching for is an assignment that satisfies each clause.

# Search Problems

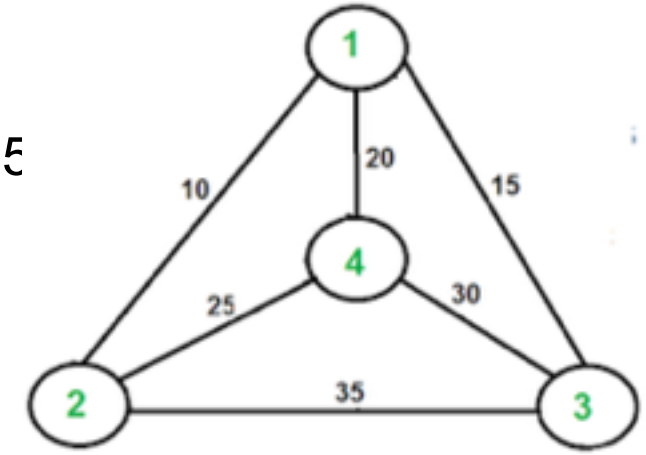
- For each such search problem, consider corresponding **checking/verifying algorithm C, which:**
  - Given inputs: an instance I and a proposed solution S
  - Runs in time polynomial in size of instance, i.e., |I|.
  - Return true if S is a solution to I, and return false if otherwise
- For SAT problem, checking/verifying algorithm C
  - take instance I, such as,

$$(x \vee y \vee \bar{z})(x \vee \bar{y})(y \vee \bar{z})(z \vee \bar{x})(\bar{x} \vee \bar{y} \vee \bar{z})$$

  - solution S, such as (x=T, y=T, z=F) ;
  - return true if S is a satisfying truth assignment for I.

# Traveling Salesman Problem

- Given n vertices 1, . . . , n, and all n(n − 1)/2 distances between them, as well as a budget b.
  - Can we tour 4 nodes with budge b=5




- Output: find a tour (a cycle that passes through every vertex exactly once) of total cost b or less – or to report that no such tour exists.
  - find permutation τ(1),...,τ(n) of vertices such that when they are toured in this order, total distance covered is at most b:
  - $d_{\tau(1),\tau(2)} + d_{\tau(2),\tau(3)} + \cdots + d_{\tau(n),\tau(1)} \leq b$.

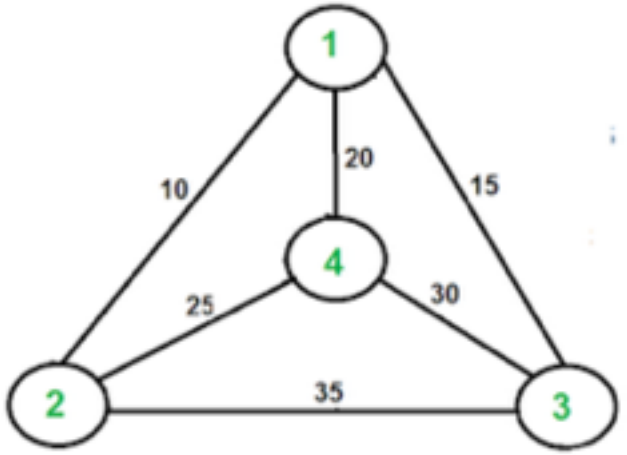
# NP Problem

For a search/decision problem, if :

- There is an efficient checking algorithm C that takes as input **the given instance I**, **the proposed solution S**, and outputs true if and only if S really is a solution to instance I; and outputs false o.w.
- Moreover running time of C(I,S) is bounded by a polynomial in |I|, the length of the instance.
- Then the search/decision problem belongs to NP, the set of search problem for which there is a polynomial time checking algorithms
  - Origin: such search problem can be solved in **polynomial** time by **nondeterministic** Turing machine

# Traveling Salesman Problem

- Given n vertices 1, . . . , n, and all n(n − 1)/2 distances between them, as well as a budget b.
- Output: find a tour (a cycle that passes through every vertex exactly once) of total cost b or less – or to report that no such tour exists.
- Here, TSP is defined as a search/decision problem
  - given an instance, find a tour within the budget (or report that none exists).
- Usually, TSP is posed as optimization problem
  - i.e., find shortest possible tour
  - 1->2->3->4, total cost: 60



# Search vs Optimization

- Turning an optimization problem into a search problem does not change its difficulty at all, because the two versions reduce to one another.
- Any algorithm that solves the optimization TSP also readily solves search problem: find the optimum tour and if it is within budget, return it; if not, there is no solution.
- Conversely, an algorithm for search problem can also be used to solve optimization problem:
  - First suppose that we somehow knew cost of optimum tour; then we could find this tour by calling algorithm for search problem, using optimum cost as the budget.
  - We can find optimum cost by binary search.

# Why Search (not Optimize)?

- Isn't any optimization problem also a search problem in the sense that we are searching for a solution that has the property of being optimal?
  - The solution to a search problem should be easy to recognize, or as we put it earlier, polynomial-time checkable.
- Given a potential solution to the TSP, it is easy to check the properties "is a tour" (just check that each vertex is visited exactly once) and "has total length ≤ b."
- But how could one check the property "is optimal"?

## Next: a collection of problems …

apparently similar problems have different complexities

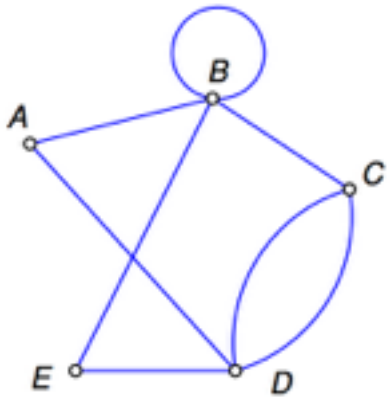
# Euler Path:

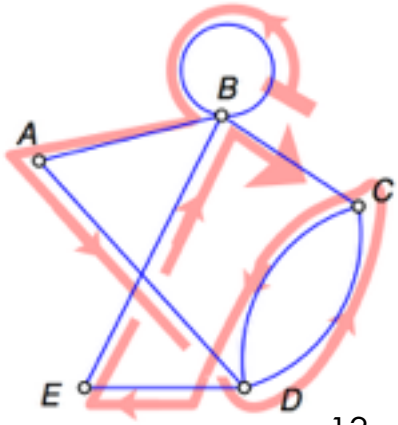
Given a graph, find a path that contains each edge exactly once.

Possible, if and only if

- (a) the graph is connected and
- (b) every vertex, with the possible exception of two vertices (the start and final vertices of the walk), has even degree.

A polynomial time algorithm for Euler Path?



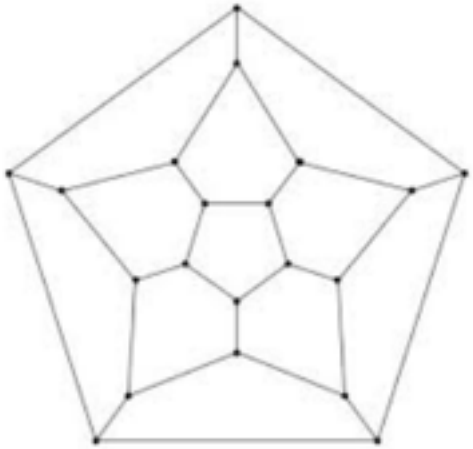



# Hamilton/Rudrata Cycle

Rudrata/Hamilton Cycle:

Given a graph, find a cycle that visits each vertex exactly once.

Recall: a cycle is a path that starts and stops at same vertex



Hamiltonian path (or traceable path) is a path in an undirected or directed graph that visits each vertex exactly once.

# Minimum Cut

A cut is a set of edges whose removal leaves a graph disconnected.

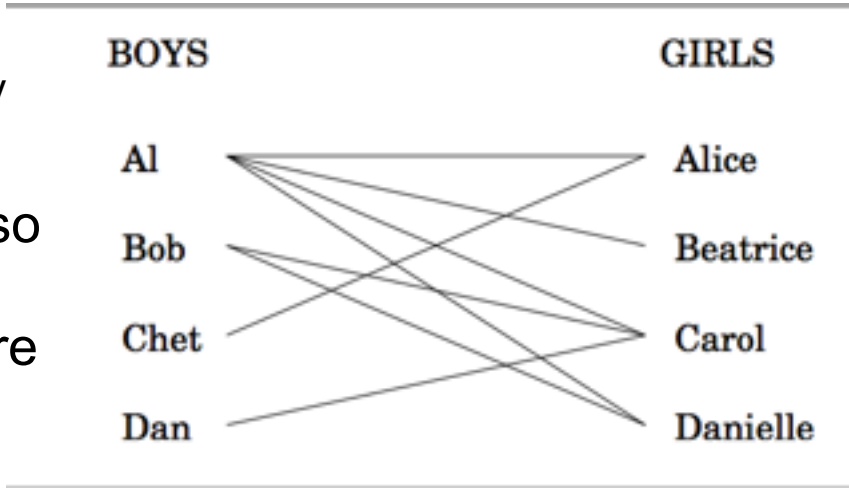
minimum cut: given a graph and a budget b, find a cut with at most b edges.

This problem can be solved in polynomial time by $n - 1$ max-flow computations: give each edge a capacity of 1, and find the maximum flow between some fixed node and every single other node.

The smallest such flow will correspond (via the max-flow min-cut theorem) to the smallest cut.

# Bipartite Matching

- Input: a (bipartite) graph
  - four nodes on left representing boys and four nodes on the right representing girls.
  - there is an edge between a boy and girl if they like each other
- Output: Is it possible to choose couples so that everyone has exactly one partner, and it is someone they like? I (i.e., is there a *perfect matching*?)
- Reduced to maximum-flow problem.
  - Create a super source node, s, with outgoing edges to all boys
  - Add a super sink node, t, with incoming edges from all girls
  - direct all edges from boy to girl, assigned cap. of 1

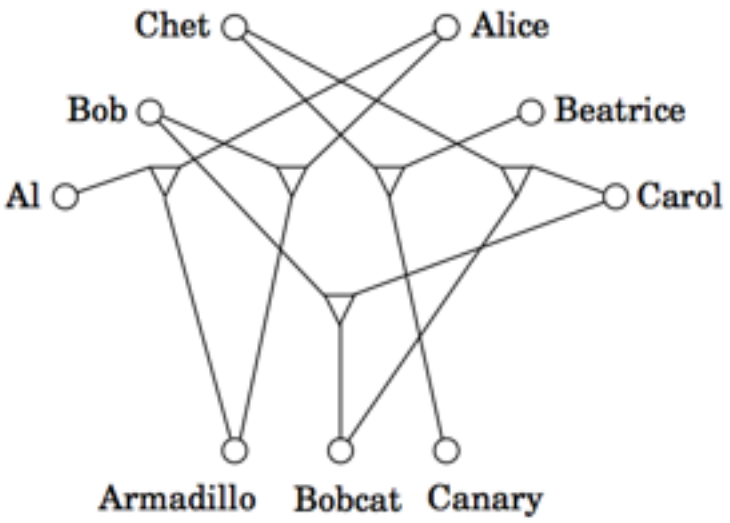


# 3D matching

3D matching: there are n boys and n girls, but also n pets, and the compatibilities among them are specified by a set of triples, each containing a boy, a girl, and a pet.

Intuitively, a triple (b,g,p) means that boy b, girl g, and pet p get along well together.

We want to find n disjoint triples and thereby create n harmonious households.



# Graph Problems

independent set:

Given a graph and an integer g, find g vertices, no two of which have an edge between them.

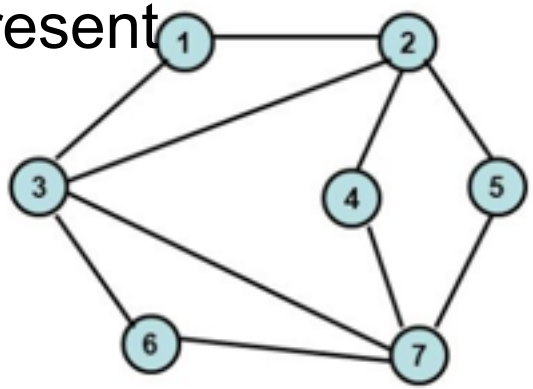
g=3, {3, 4, 5}

vertex cover: Given a graph and an integer b, find b vertices cover (touch) every edge.

b=1, no solution; b=2, {3, 7}

Clique: Given a graph and an integer g, find g vertices such that all possible edges between them are present.

g=3, {1, 2, 3}.



# Knapsack

knapsack: We are given integer weights $w_1, \ldots, w_n$ and integer values $v_1,\ldots,v_n$ for n items.

We are also given a weight capacity W and a goal g

We seek a set of items whose total weight is at most W and whose total value is at least g.

The problem is solvable in time O(nW) by dynamic programming.

subset sum:
Find a subset of a given set of integers that adds up to exactly W .

# Hard Problems, Easy Problems

| Hard problems (**NP**-complete) | Easy problems (in **P**) |
|---|---|
| 3SAT | 2SAT, HORN SAT |
| TRAVELING SALESMAN PROBLEM | MINIMUM SPANNING TREE |
| LONGEST PATH | SHORTEST PATH |
| 3D MATCHING | BIPARTITE MATCHING |
| KNAPSACK | UNARY KNAPSACK |
| INDEPENDENT SET | INDEPENDENT SET on trees |
| INTEGER LINEAR PROGRAMMING | LINEAR PROGRAMMING |
| RUDRATA PATH | EULER PATH |
| BALANCED CUT | MINIMUM CUT |

# P

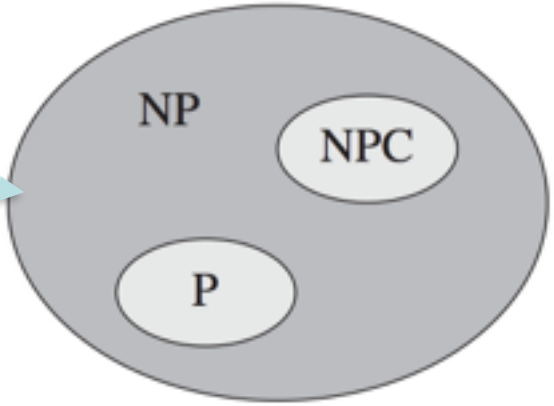
We've seen many examples of NP search problems that are solvable in polynomial time.

In such cases, there is an algorithm that takes as input an instance I and has a running time polynomial in |I|. If I has a solution, the algorithm returns such a solution; and if I has no solution, the algorithm correctly reports so.

The class of all search problems that can be solved in polynomial time is denoted P.

# P=NP?

- Most people believe not



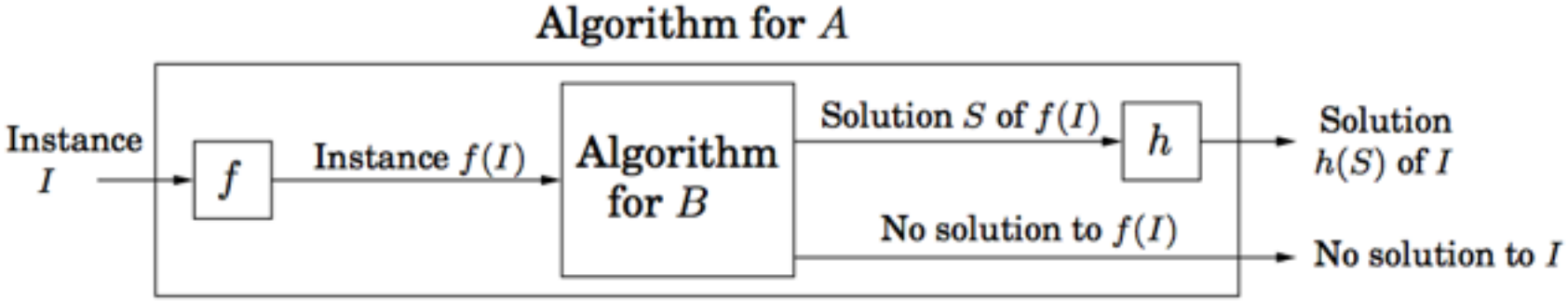

- Many problems have no polynomial time algorithms … yet.
- All problems on left side of table are same problem.
  - If one of them has a polynomial time algorithm, then every problem has a polynomial time algorithm.
  - **NP Complete (NPC)**

# Reduce A -> B

A reduction from search problem A to search problem B

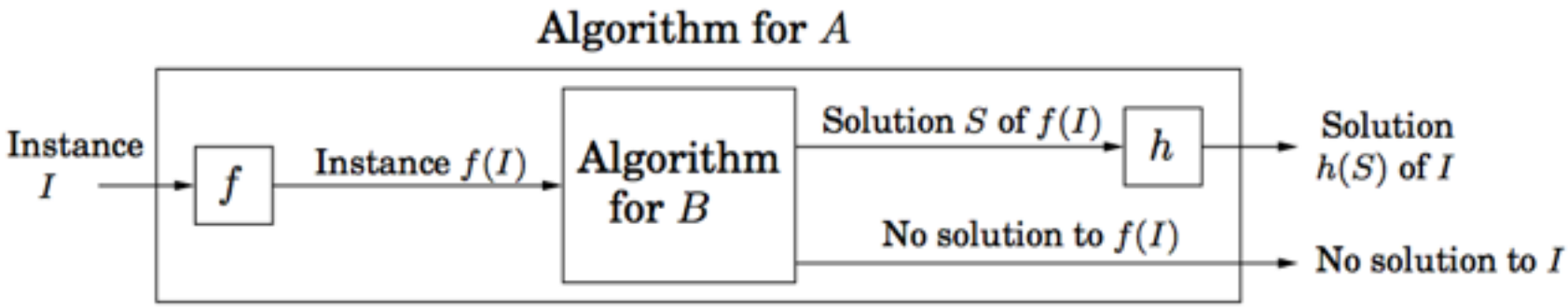
- a polynomial time algorithm **f** that transforms any instance I of A into an instance f(I) of B
- and another polynomial time algorithm **h** that maps any solution S of f(I) back into a solution h(S) of I.
- If f (I ) has no solution, then neither does I .

Any algorithm for B can be converted into an algorithm for A by bracketing it between f and h.



# Reduction

- Assume there is a reduction from a problem A to a problem B. A → B.
  - If we can solve B efficiently, then we can also solve A efficiently.
  - If we know A is hard, then B must be hard too.
    Reductions also have the convenient property that they compose. If A → B and B → C, then A → C.



# NP Complete

Definition:

A search problem C is NP-complete

1) It's NP

2) Every NP problem can be reduced to C.

# 3SAT -> Independent Set

**3SAT (special case of SAT)**

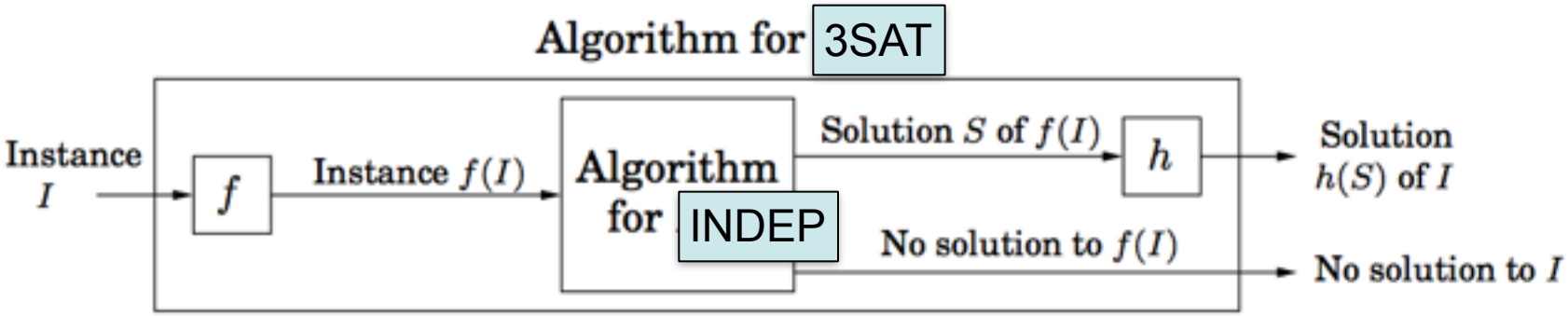
input: a set of clauses, each with three or fewer literals,

Output: a satisfying truth assignment (if exists)

**Independent Set**

Input: a graph and a number g

Output: a set of g pairwise non-adjacent vertices (if exists)



# 3SAT -> Independent Set

**3SAT:** find satisfying truth assignment for a set of clauses

$$(\overline{x} \vee y \vee \overline{z})\ (x \vee \overline{y} \vee z)\ (x \vee y \vee z)\ (\overline{x} \vee \overline{y}),$$

**Independent Set** input: a graph and a number g

Output: find a set of g pairwise non-adjacent vertices.

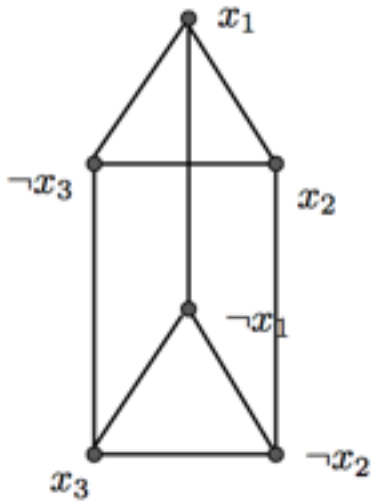
Given an instance I of 3SAT, create an instance (G,g) of Independent Set as follows:

- Graph G has a triangle for each clause (or just an edge, if the clause has two literals), with vertices labeled by the clause's literals, edges between any two vertices that represent opposite literals.
- Goal g is set to the number of clauses.

# 3SAT -> Independent Set

**3SAT:** find satisfying truth assignment for a set of clauses

$$(x_1 \vee x_2 \vee \neg x_3) \wedge (x_3 \vee \neg x_2 \vee \neg x_1)$$




Find independent set of size 2

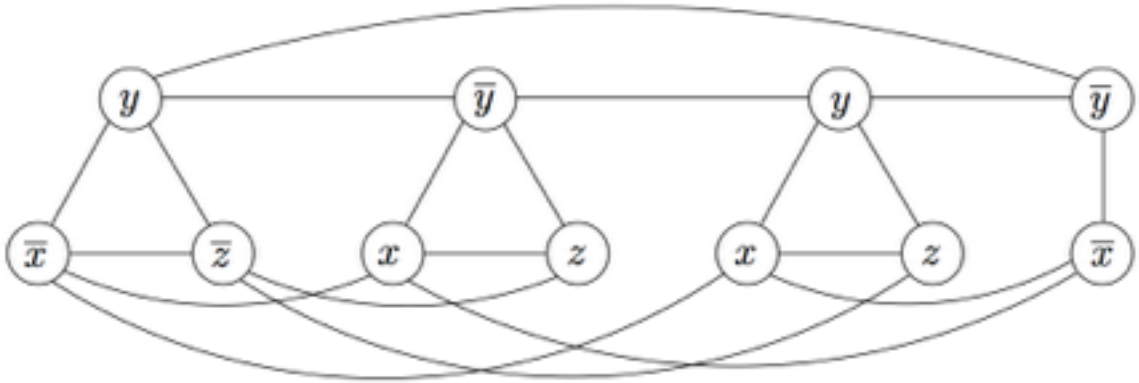
**Independent Set** input: a graph and a number g

Output: find a set of g pairwise non-adjacent vertices.

# 3SAT -> Independent Set

**3SAT:** find satisfying truth assignment for a set of clauses

**Figure 8.8** The graph corresponding to $(\overline{x} \vee y \vee \overline{z})\ (x \vee \overline{y} \vee z)\ (x \vee y \vee z)\ (\overline{x} \vee \overline{y})$.




Find independent set of size 4

**Independent Set** input: a graph and a number g

Output: find a set of g pairwise non-adjacent vertices.

# SAT -> 3SAT

Given an **instance I** of SAT where clauses have more than three literals, $(a_1 \vee a_2 \vee \cdots \vee a_k)$ ($a_i$ 's are literals, k > 3), is replaced by a set of clauses.

$$(a_1 \vee a_2 \vee y_1)(\bar{y}_1 \vee a_3 \vee y_2)(\bar{y}_2 \vee a_4 \vee y_3) \cdots (\bar{y}_{k-3} \vee a_{k-1} \vee a_k),$$

where $y_i$ 's are new variables.

The conversion takes polynomial time.

Resulting CNF, I′, is equivalent to I in terms of satisfiability, because for any assignment to the $a_i$ 's,

$$\left\{ \begin{array}{c} (a_1 \vee a_2 \vee \cdots \vee a_k) \\ \text{is satisfied} \end{array} \right\} \iff \left\{ \begin{array}{c} \text{there is a setting for the } y_i\text{'s for which} \\ (a_1 \vee a_2 \vee y_1)(\bar{y}_1 \vee a_3 \vee y_2)(\bar{y}_2 \vee a_4 \vee y_3) \cdots (\bar{y}_{k-3} \vee a_{k-1} \vee a_k) \\ \text{are all satisfied} \end{array} \right\}$$

# Summary