Relations

CISC1100, Fall 2009

Overview: relations

- Binary relations
  - Defined as a set of ordered pairs
  - Graph representations
- Properties of relations
  - Reflexive, Irreflexive
  - Symmetric, Anti-symmetric
  - Transitive

Relations

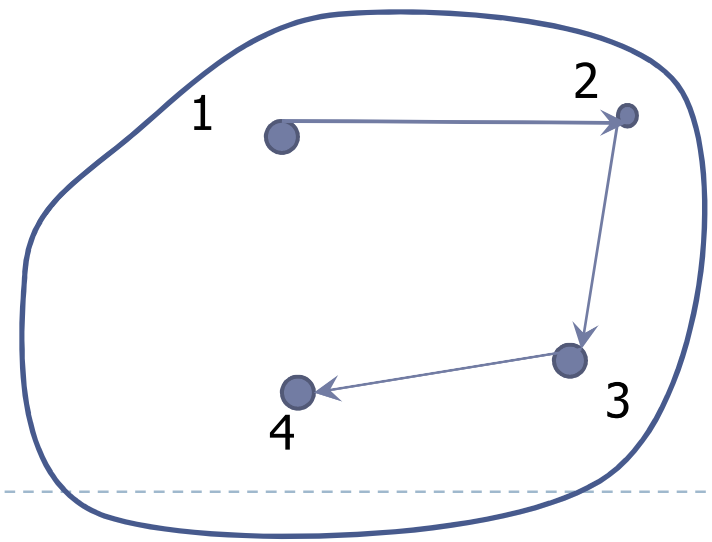
- Two people are related, if there is some family connection between them
- We study more general relations:
  - "is the same major as" is a relation defined among all college students
    - If Jack is the same major as Mary, we say Jack is related to Mary under "is the same major as" relation
    - This relation goes both way, i.e., symmetric
  - "is older than" defined among a set of people
    - This relation does not go both way
  - " is facebook friend with", …

Relations between numbers

- Comparison relation
  - =, <, >, <=, …
- Other relations
  - Add up to 10, e.g., 2 and 8 is related under this relation, and so is 5 and 5, …
  - Is divisible by
    - a is divisible by b, if a can be expressed as b*c, where c is an integer
    - E.g. 6 is divisible by 2, 5 is not divisible by 2, 5 is divisible by 5, …

Using graph to visualize graph

- A cloud where everything falls within
  - nodes (solid small circle): people, numbers, …
  - Arcs: connecting two people, numbers that are related
  - Arrows: the direction of the "relation"…



1 is related to 2,
2 is related to 3.
3 is related to 4

Relations between sets

- Given some sets, {},{1}, {2}, {1,2}, {1,2,3}
  - "Is a subset of" relation:
    - {} is a subset of {1}
    - {1} is a subset of {1,2}, …
  - "Has more elements than" relation:
    - {1} has more elements than {}, …
  - "Have no common elements with" relation:
    - {} has no common elements with {1},
    - {1} has no common elements with {2}…
  - Practice: draw the graph for each of above relations

Binary relations

- Relations is defined on a collection of people, numbers, sets, …
  - We refer to the set (of people, numbers, …) as the domain of the relation
  - A rule specifies the ordered pair of objects in S that are related
- So far, the rule has been given in English
- Generally, we use $R_{subscript}$ to denote a relation
  - $R_{is\_older\_than}$
  - $R_{one\_more\_than}$

Ways to describe a binary relation

- Consider domain S={1,2,3}, and the "smaller than" relation, $R_<$
- In English, we say "a is related to b, if a is smaller than b". (i.e., specifying the rule)
- We can just list all pairs that are related, i.e., 1 is smaller than 2, 1 is smaller than 3, 2 is smaller than 3.
  - More concisely, (1,2),(1,3),(2,3) are all ordered pairs of elements that are related under $R_<$
  - $R_<$={(1,2), (1,3),(2,3)}

Formal definition of binary relation

- For the domain S, the set of all possible ordered pairs of elements from S is the cartesian product, S x S.
- Def: a binary relation R defined on domain S is a subset of S x S
- For example: S={1,2,3}, below are relations on S
  - $R_1=\{(1,2)\}$
  - $R_2=\{\}$, no number is related to another number
  - $R_3 = \{(a,b) \mid a \in S \text{ and } b \in S \text{ and } a+b > 2\}$
  - R_4=S x S, i.e., all ordered pairs are related under R_4

Define binary relation as set (cont'd)

- Sometimes relation is between two different kinds of objects
  - "goes to college at" relation is defined from the set of college students, to the set of colleges
- Given two sets S and T, a binary relation from S to T is a subset of SxT.
  - S is called domain of the relation
  - T is called codomain of the relation
- We focus on binary relation with same domain and codomain for now.

Binary relation: infix notation

- For example, S={1,2,3}, followings are relations on S
  - $R_<$={(1,2),(2,3),(1,3)}
- We say 1 and 2 are related under $R_<$, or just "1 $R_<$ 2". For this relation, we have special symbol, "1 < 2".
- Ex:
  - 1 R_4 2, 2 R_4 2, …
  - But 1 R_4 1 is false

$$R_4 = \{(a,b) \mid a \in S \text{ and b} \in \text{S and a} + \text{b} > 2\}$$

Domain can be infinite set

Domain: Z

R: {(a, b) is an element of Z x Z : (a - b) is even}

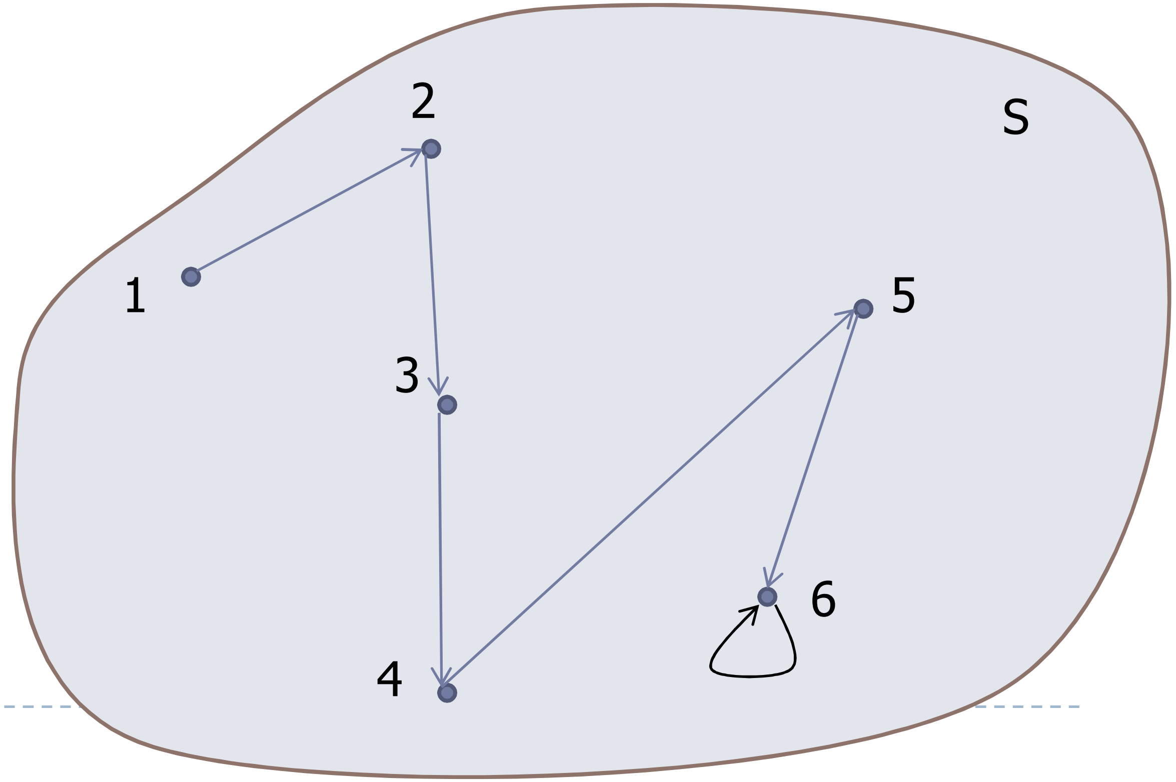
Given any pair of integers a, b, we can test if they are related under R by checking if a-b is even

e.g., as 5-3=2 is even, 5 is related to 3, or $(5,3) \in R$

e.g., as 5-4 is odd, $(5,4) \notin R$

Relations can be defined using graph

- S={1,2,3,4,5,6}
- R={(1,2), (2,3),(3,4), (4,5),(5,6), (6,6)}



Some exercises

- For each of following relations defined on set {1,2,3,4,5,6}, write set enumeration of the relation, and draw a graph representation:
  - $R_{\leq}$: "smaller or equal to"
  - R_d: "divides": e.g., 2 divides 6
  - R_a: "adds up to 6", e.g., (3,3), (1,5) …

Operations on relation

- Relations is a set of ordered pairs
  - Set operations can be applied to relations
- Ex. Let R_1 and R_2 be two relations on N:
  - $(x, y) \in R_1$ if and only if x=y;
  - $(x, y) \in R_2$ if and only if x<y
- What are the following relations?

$$R_1 \cup R_2 \qquad R_1 \cap R_2$$

$$R_1^{\,c}$$

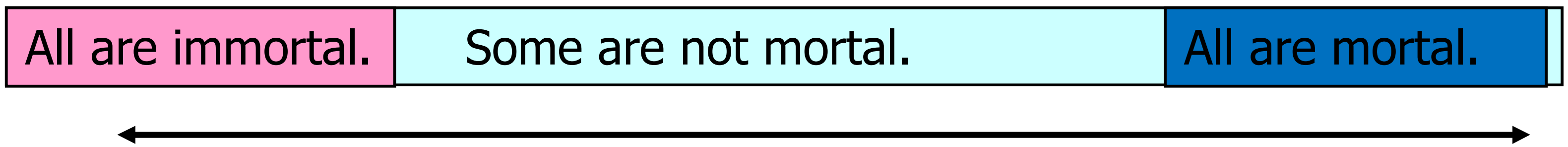
$$R_2^{\,c}$$

Relationships have properties

- Properties of relations:
  - Reflexive, irreflexive
  - Symmetric, Anti-symmetric
  - Transitive
- We will introduce the definition of each property and learn to test if a relation has the above properties

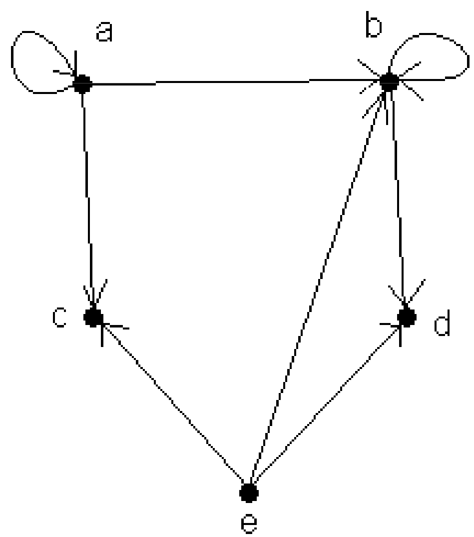
Primer about negation

- Let's look at a statement that asserts something about all human being:
  - All human beings are mortal. (a)
- The opposite of above statement:
  - All human beings are immortal. (b)
- The negation of above statement:
  - It's not true that "all human beings are mortal"
  - i.e., Some human beings are not mortal.



Reflexive Property

- Def: A relation is reflexive if all elements in the domain are related to themselves
- If R is reflexive, there is a loop on each node in its graph
- A relation is not reflexive if there is some element in the domain that is not related to itself



Not reflexive since e does not go back to e

Reflexive Property

- For example
  - If our set consists of all people,
  - And the relation is "Is the same age as"
- Well we pick several person and see what happens
  - Tom is the same age as Tom
  - Carol is the same age as Carol
  - Sally is the same age as Sally
- It always work, i.e., for any person, he/she is the same age as himself/herself.
- Thus "is the same age as" is reflexive

Try this mathematical one

- Domain is Z
- R={(x, y) | x,y $\in$ Z, and (x + y) is an even number}
- Is R reflexive?
  - Is any number in Z related to itself under R ?
  - Try a few numbers, 1, 2, 3, …
  - For any numbers in Z ?
  - Yes, since a number added to itself is always even (since 2 will be a factor), so R is reflexive

Another example

- Consider the set R (all real numbers) as domain
- Consider "is larger than" relation
- Try a few examples:
  - Pick a value 5 and ask "Is 5 larger than 5" ?
  - No, i.e., 5 is not related to itself
- Therefore, this relation is NOT Reflexive
- Actually, no real number is larger than itself, i.e., NOT related to itself
  - Irreflexive relation

Irreflexive Relation

- For some relations, no element in the domain is related to itself.
  - $R_>$ defined on R (the set of all real numbers)
    - 1 is not related to itself under this relation, neither is 2 and 3 related to itself
  - "is older than" relation defined on a set of people
- Def: a relation R on domain A is irreflexive if for all $a \in A$, (a, a) is not an element of R
- An irreflexive relation's graph has no self-loop

For all relations

No element is related to itself

Some element is related to itself, some element is not related to itself

Every element is related to itself

irreflexive	Not reflexive, not irreflexive	Reflexive

All relations

A relation cannot be both reflexive and irreflexive.

Exercises: reflexive? irreflexive ?

- Each of following relations is defined on set {1,2,3,4,5,6},
  - $R_{\leq}$:"smaller or equal to"
  - R_d: "divides": e.g., 2 divides 6, but 4 doesn't divides 6
  - R_a: "adds up to 6", e.g., (3,3), (1,5) …
  - R={(1,2),(3,4), (1,1)}

Symmetric Property

- some relations are mutual, i.e., works both ways, we call them symmetric
- E.g., "has the same hair color as" relation among a set of people
  - Pick any two people, say A and B
  - If A has the same color hair as B, then of course B has the same color hair as A
  - Thus it is symmetric
  - Other example, "is a friend of", "is the same age of", "goes to same college as"
- In the graphs of symmetric relations, arcs go both ways (with two arrows)

Exercise

- Domain: {1, 2, 3 ,4}
- Relation={(1, 2), (1, 3), (4, 4), (4, 5), (3, 1), (5,4), (2, 1)}
- Yes, it is symmetric since
  - (1,2) and (2,1)
  - (1,3) and (3,1)
  - (4,5) and (5,4)

A relation that is not symmetric

- Some relations doesn't go both ways
  - For example, "is older than" relation
  - If "Sally is older than Tom", then "Tom is older than Sally" must be false
  - We found a case where (Sally, Tom) is in the relation, but (Tom, Sally) is not in the relation
- Therefore, this relation is not symmetric.
- Actually, for "is older than" relation, it never works both way
  - Anti-symmetric relation

Anti-symmetric Property

- Some relations never goes both way
  - E.g. "is older than" relation among set of people
  - For any two persons, A and B, if A is older than B, then B is not older than A
  - i.e., the relation **never** goes two ways
- Such relations are called anti-symmetric relations
- In the graph, anti-symmetric relations do not have two-way arcs.

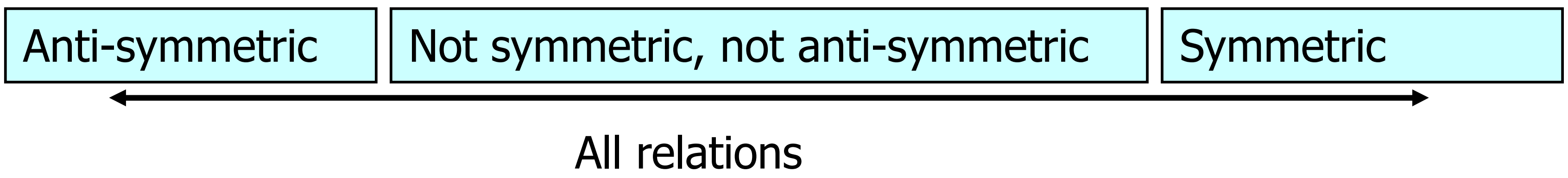
Formal Definition of Symmetric/Anti-symmetric

A relation R defined on A is symmetric if for all $a, b \in A$, if $(a, b) \in R$, then $(b, a) \in R$

A relation R defined on A is anti-symmetric if for all $a, b \in A$, if $a \neq b$ and $(a, b) \in R$ then $(b, a) \notin R$

Bizarre Middle Ground

- Some relations are symmetric
- Some relations are anti-symmetric
- Some relations are not symmetric, and not anti-symmetric, i.e.,
  - Some ordered pairs can be reversed
  - Some ordered pairs cannot be reversed



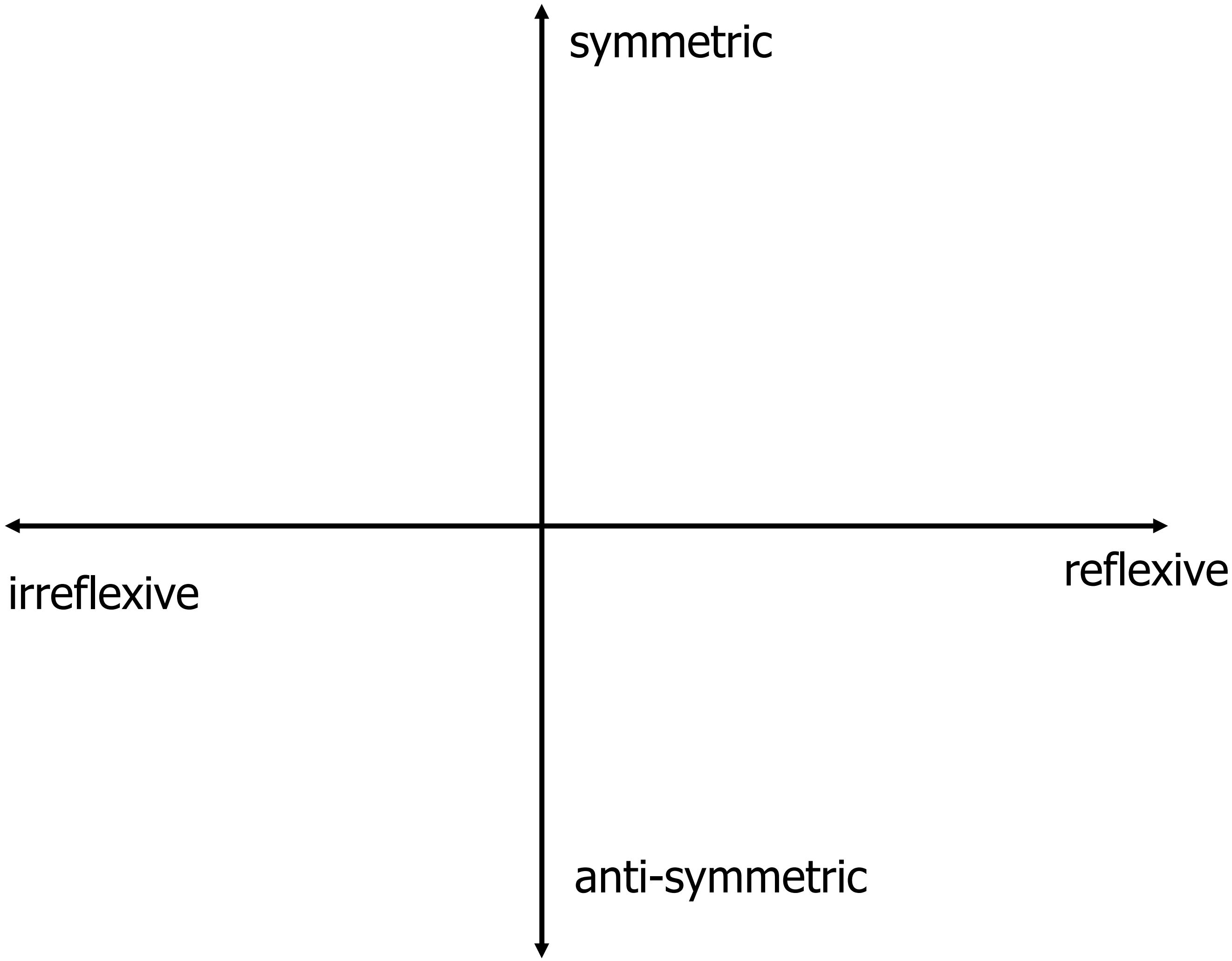
Ex: a relation that is neither

- Consider "knows the birthday of" defined among a set of people
- Some pair of people know each other's birthday,
  - Tom know the birthday of Sally
  - Sally know the birthday of Tom
  - So we now know that it is not anti-symmetric.
- Some people know the birthday of the other, but not vice versa
  - Tom knows the birthday of Bill
  - Bill doesn't know the birthday of Tom
  - So the relation is not symmetric.
- Therefore this relation is neither symmetric nor anti-symmetric

Exercises: Symmetric? Anti-symmetric ?

- For each of following relations defined on set {1,2,3,4,5,6}
  - R={(1,2),(3,4), (1,1),(2,1),(4,3)}
  - R={(1,2),(3,4),(1,1)}
  - R={(1,2),(3,4),(1,1),(4,3)}
  - $R_{\leq}$:"smaller or equal to"
  - R_d: "divides": e.g., 6 divides 2
  - R_a: "adds up to 6", e.g., (3,3), (1,5) …

So far, the picture of relations



Transitive: an introduction

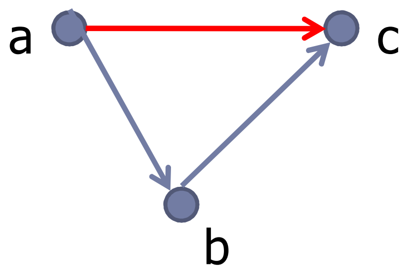
- You are assigned a job to draw graph that represents "is older than" relation defined on our class
  - You can only ask questions such as "Is Alice older than Bob?"
  - You want to ask as few questions as possible.
- Suppose you already find out:
  - Alice is older than bob. i.e., (Alice, Bob) $\in R_{is\_older}$
  - Bob is older than Cathy. i.e., (Bob, Cathy) $\in R_{is\_older}$
- Do you need to ask "Is Alice older than Cathy?"
  - No !
  - Alice for sure is older than Cathy. We know that for any three people, A, B and C, if A is older than B, B is older than C, then fore sure, A is older than C.
- Such property of this relation is called transitive.

Is this relation transitive ?

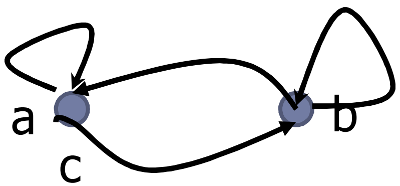
- "is taking the same class as" relation on a set of students
- Suppose three students, Bob, Katie, and Alex,
  - Bob is taking the same class as Katie
  - Katie is taking the same class as Alex
- And now consider:
  - Is Bob is taking the same class as Alex ?
- Many cases: no
  - Katie takes 1100 with Bob, and takes history with Alex, where Bob and Alex has no classes in common.
- Therefore this relation is not transitive

Transitive Property

- A relation R is transitive if for any three elements in the domain, a, b, and c, knowing that a is related to b, and b is related to c would allow us to infer that a is related to c.



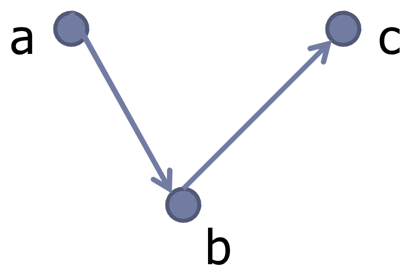
In graph of a transitive relation:
if there is two-hop paths from a to c,
then there is one-hop path from a to c.



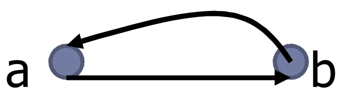
- E.g. "is older than", "is same age as" is transitive

Not Transitive

- A relation R is not transitive if there exists three elements in the domain, a, b, and c, and a is related to b, b is related to c, but a is not related to c.



In graph: there is two-hop paths from a to c, but there is not a one-hop path from a to c.



- E.g. "is taking same class as", "know birthday of"

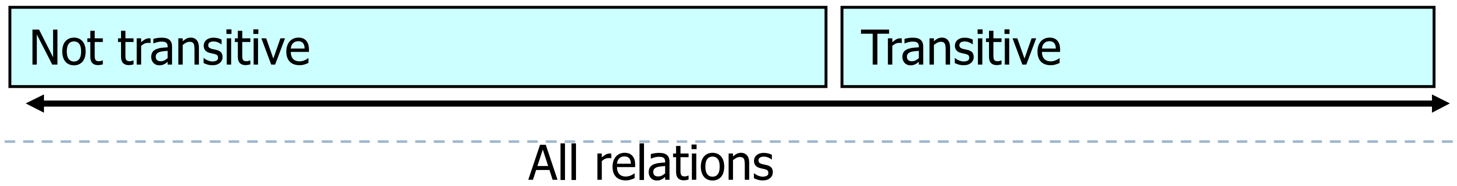
Formal Definition of Transitive

Relation R on domain A is transitive, if

for any $a, b, c \in A$,

if $(a,b) \in R$ and $(b,c) \in R$,

then $(a,c) \in R$



Exercises: Transitive or not ?

- $R_{\leq}$:"smaller or equal to" defined on set {1,2,3,4,5,6}
  - For three numbers a, b, c from {1,2,3,4,5,6}
    - Would knowing that a≤b, and b≤c, allows me to conclude that a ≤c ?
    - Yes !
  - It's transitive !
  - Let's check it's graph …

Exercises: Transitive or not ?

- Are the following relations defined on set {1,2,3,4,5,6} transitive ?
  - R={(1,2),(3,4), (1,1),(2,1),(4,3)}
  - R={(1,2),(3,4),(1,1)}
  - R={(1,2),(2,4),(1,1),(4,3)}
  - $R_{\leq}$:"smaller or equal to"
  - R_d: "divides": e.g., 6 divides 2
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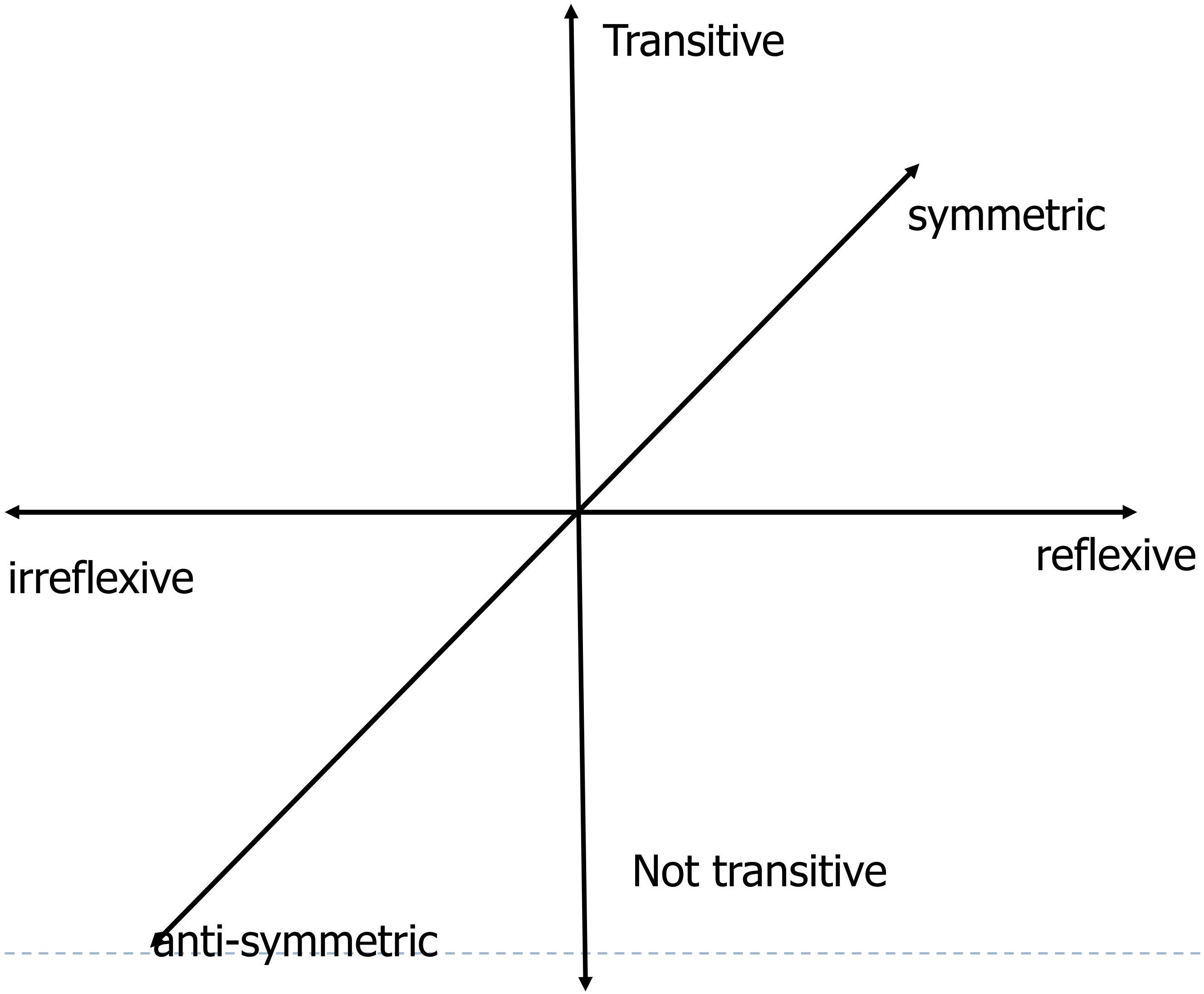

What properties does it have?

- Domain: (1, 2, 3, 4, 5)
- Codomain: (1, 2, 3, 4, 5)
- Relation={(1, 1), (1, 2), (1, 3), (2, 3), (2, 4), (3, 4), (5,5)}
- The Answer is:
  - It is not reflexive, and it is not irrefleixve
  - It is not symmetric, and it is anti-symmetric
  - It is not transitive

What properties does it have?

- Domain: (1, 2, 3, 4, 5)
- Codomain: (1, 2, 3, 4, 5)
- Relation={(1, 1), (2, 2), (3, 3), (4, 4), (5, 5)}
- The Answer is:
  - It is reflexive, and it's not irreflexive
  - It is symmetric, and anti-symmetric
  - It is transitive

Exercise

- Comment on the properties of the relation
  - "Subset" relation R on the set P({1,2,3}) with the rule:

$$(x, y) \in R \text{ if } x \subseteq y$$

Exercise

- For the following relations defined on A={1,2,3,4,5,6,7,8}
  - R={(1,1),(1,2),(1,4),(1,8),(2,2),(2,4),(2,8),(3,3),(3,6),(4,4),(4,8),(5, 5),(6,6),(7,7),(8,8)}

Try a few out

- Consider all positive numbers. Identify properties for the following relations
  - R1: "Is less than or equal to"
  - R2: "Is a factor of"
  - R3: "Is 10 less than"

Try a few out

- Consider following relations defined on the set of all people. Identify their properties:
  - R_1:"Is smarter than"
  - R_2:"Went to the same high school as"
  - R_3:"Is a cousin of"

Partial Ordering & Topological Sorting

- A relation R on a set S that is reflexive, anti-symmetric, and transitive is called a partial ordering on S.
  - e.g. "less than or equal to", "is a subset of", "is prerequisite of"
  - If (a,b) is related under R, we call a is predecessor of b, and b is a successor of a.
- Topological sorting: given a partial ordering on S, order elements in S such that all predecessors appear before their successors.

The idea of algorithm

- Input:
- Output:
- Step by step procedure: