CISC 1100: Structures of Computer Science

Chapter 3
Logic

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Logical (or illogical?) reasoning

- Is this a valid argument?
  - All men are mortal.
  - Socrates is a man.
  - Therefore,

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  Valid or not? Yes!
- Is this a valid argument?
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  - We are going out for ice cream.
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  Valid or not? No!
- How to recognize the difference?

Outline

- Propositional logic
  - Logical operations
  - Propositional forms
  - From English to propositions
  - Propositional equivalence
- Predicate logic
  - Quantifiers
  - Some rules for using predicates

Propositional logic

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- Colorless green ideas sleep furiously.

Propositional logic (cont'd)

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- Propositional variables: lower case letters (p, q, ...)
  (Analogous to variables in algebra.)
  - $p=$"A New York City subway fare is \$2.50."
  - $q=$"It will rain today in Manhattan."
  - $r=$"All multiples of four are even numbers."

Logical operations: negation

- *Negation*, the NOT operation: reverses a truth value.
- Negation is a *unary operation*: only depends on one variable.
- Negation of p is denoted p'.
  (Some books use other notations, such as $\overline{p}$, $\sim p$, or $\neg p$.)
- Can display via a *truth table*

p	p'
T	F
F	T

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- Truth tables:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
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Logical operations: exclusive or

- The inclusive or $\vee$ is not the "or" of common language.
- That role is played by *exclusive or* (XOR), denoted $\oplus$.
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- Be careful to distinguish between $\textsc{or}$ and $\textsc{xor}$!

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  "One can derive anything from a false hypothesis."

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Systematize the process via a *parse tree*.

Parse tree for $-(1+2)/(3 \times 4) + (5 + 6 \times 7)/(8+9) - 10$:

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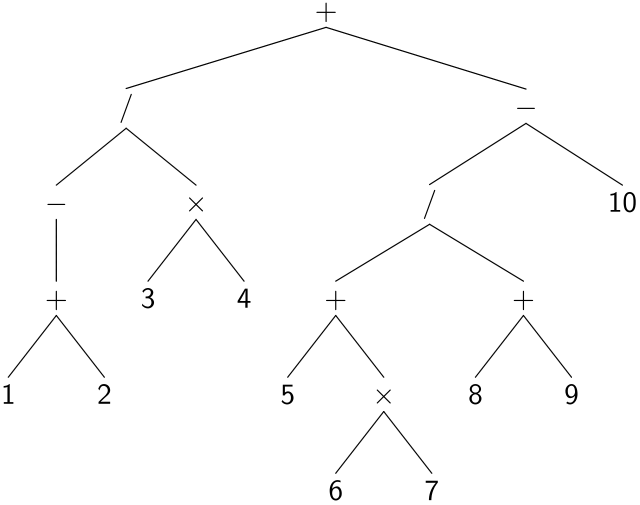

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We're inherently using the following rules:

1. Parenthesized subexpressions are evaluated first.
2. Operations have a *precedence hierarchy*:
   1. Unary operations (for example, -1) are done first.
   2. Multiplicative operations ($\times$ and $/$) are done next.
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These guarantee that (e.g.) $2 + 3 \times 4$ is 14, rather than 20.

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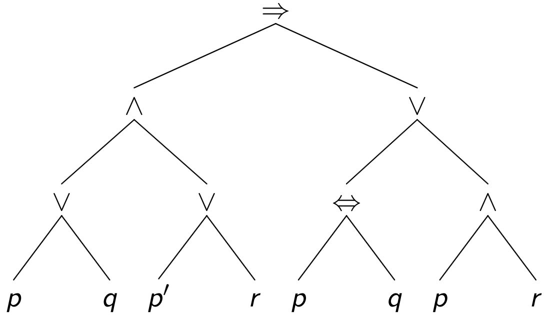

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  1. Parenthesized subexpressions are evaluated first.
  2. Operations have a *precedence hierarchy*:
     1. Unary negations ($'$) are done first.
     2. Multiplicative operations ($\wedge$) are done next.
     3. Additive operations ($\vee$, $\oplus$) are done next.
     4. The conditional-type operations ($\Rightarrow$ and $\Leftrightarrow$) are done last.
  3. In case of a tie (two operations at the same level in the hierarchy), operations are done in a left-to-right order, *except* for the conditional operator $\Rightarrow$, which is done in a right-to-left order. That is, $p \Rightarrow q \Rightarrow r$ is interpreted as $p \Rightarrow (q \Rightarrow r)$.

Propositional Forms (cont'd)

So can replace

$$[(p \vee q) \wedge ((p') \vee r)] \Rightarrow [(p \Leftrightarrow q) \vee (p \wedge r)]$$

by

$$[(p \vee q) \wedge (p' \vee r)] \Rightarrow [(p \Leftrightarrow q) \vee p \wedge r]$$

or even

$$(p \vee q) \wedge (p' \vee r) \Rightarrow (p \Leftrightarrow q) \vee p \wedge r.$$

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Propositional Forms (cont'd)

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- Let's simplify!
  1. Parenthesized subexpressions come first.
  2. Next comes the only unary operation ($'$).
  3. Next comes the only multiplicative operation ($\wedge$).
  4. Next comes the additive operations ($\vee$, $\oplus$).
  5. Use parentheses if you have *any* doubt.
     Always use parentheses if you have multiple conditionals.
  6. Evaluate ties left-to-right.

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  Solution? $p \Rightarrow c$.

- **Example:** If Alice will have coffee and Bob will go to the beach, then either Carol will be disappointed or I will make peanut butter sandwiches.
  Solution?

From English to Propositions (cont'd)

- **Example:** If Alice will have coffee and Bob will go to the beach, then either Carol will be disappointed or I will make peanut butter sandwiches.
  Solution? $a \wedge b \Rightarrow c \vee p$

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  Alice will have coffee and
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  if and only if

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  Solution? $(a \wedge b') \Leftrightarrow (c \wedge p')$

Propositional Equivalence

High school algebra: establishes many useful rules, such as

$$\begin{aligned} a + b &= b + a, \\ a \times (b + c) &= a \times b + a \times c, \\ -(a + b) &= (-a) + (-b), \end{aligned}$$

Anything analogous for propositions?

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Anything analogous for propositions?

- How to state them? (No equal sign.)
- How to prove correct rules?
- How to disprove incorrect "rules"?

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- Beware!
  - $p \equiv q$ is *not* a proposition; it's a statement *about* propositions.
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we might *conjecture* that
$$\begin{aligned} p \vee q &\equiv q \vee p, \\ p \wedge (q \vee r) &\equiv (p \wedge q) \vee (p \wedge r), \\ (p \vee q)' &\equiv p' \vee q'. \end{aligned}$$

- Want to prove (or disprove) conjectured identities such as

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\end{gathered}
$$

- How? Use a truth table.
- Suppose that p and q are propositional formulas.
  The equivalence $p \equiv q$ is true iff the truth tables for p and q are identical.

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Propositional Equivalence (cont'd)

Example: Is it true that $p \lor q \equiv q \lor p$?

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T	T	T
T	F	T
F	T	T
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p	q	$q \lor p$
T	T	T
T	F	T
F	T	T
F	F	F

They match! So $p \lor q \equiv q \lor p$.
More compact form:

p	q	$p \lor q$	$q \lor p$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F

Example: Is it true that $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$?

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p	q	r	$q \vee r$	$p \wedge (q \vee r)$	$p \wedge q$	$p \wedge r$	$(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

Propositional Equivalence (cont'd)

Example: Is it true that $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$?

p	q	r	$q \vee r$	$p \wedge (q \vee r)$	$p \wedge q$	$p \wedge r$	$(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
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So $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$.

Propositional Equivalence (cont'd)

- How to organize the table?
  - Two variables: TT, TF, FT, FF
  - Three variables: TTT, TTF, TFT, TFF, FTT, FTF, FFT, FFF.
  - General pattern?
    - Rightmost variable alternates: TFTFTFTF ...
    - Next alternates in pairs: TTFFTTFF ...
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  - n variables? 2^n rows.
  - Since $2^{10} = 1024$, you don't want to do a 10-variable table.

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T	F	T	F	F	T	T
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So it is *not* true that $(p \vee q)' \equiv p' \vee q'$!

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F	T	T	F	T	F	F
F	F	F	T	T	T	T

The formula $(p \wedge q)' \equiv p' \vee q'$ is also correct.
These formulas

$$(p \vee q)' \equiv p' \wedge q'$$
$$(p \wedge q)' \equiv p' \vee q'$$

are called *deMorgan's laws*.

Propositional Equivalence (cont'd)

Some well-known propositional laws (we haven't proved them all):

Double Negation	$(p')' \equiv p$
Idempotent	$p \wedge p \equiv p$
Idempotent	$p \vee p \equiv p$
Commutative	$p \wedge q \equiv q \wedge p$
Commutative	$p \vee q \equiv q \vee p$
Associative	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
Associative	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
Distributive	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
Distributive	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
DeMorgan	$(p \wedge q)' \equiv (p') \vee (q')$
DeMorgan	$(p \vee q)' \equiv (p') \wedge (q')$
Modus Ponens	$[(p \Rightarrow q) \wedge p] \Rightarrow q$
Modus Tollens	$[(p \Rightarrow q) \wedge q'] \Rightarrow p'$
Contrapositive	$(p \Rightarrow q) \equiv (q' \Rightarrow p')$
Implication	$(p \Rightarrow q) \equiv (p' \vee q)$

Propositional Equivalence (cont'd)

The preceding table is similar to the table of set identities from Chapter 1, e.g., we have

$$(p \wedge q)' \equiv p' \vee q' \qquad \text{and} \qquad (A \cap B)' = A' \cup B' \,.$$

It turns out that we can use a propositional law to easily prove the analogous set identity.

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Example: Show that $(A \cap B)' = A' \cup B'$.

Solution: Must show that any element of $(A \cap B)'$ is an element of $A' \cup B'$, and vice versa. But

$$x \in (A \cap B)' \iff (x \in A \cap B)'$$

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as required. $\square$

Once we've proved a given propositional law, we can use it to help prove new ones.

Example: Let's prove the *exportation identity*

$$[(p \wedge q) \Rightarrow r] \equiv [p \Rightarrow (q \Rightarrow r)].$$

We have

$$(p \wedge q) \Rightarrow r \equiv (p \wedge q)' \vee r \qquad \text{implication}$$

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Propositional Equivalence (cont'd)

- **Duality:** If p is a proposition that only uses the operations $'$, $\wedge$, and $\vee$. If we replace all instances of $\wedge$, $\vee$, T, and F in p by $\vee$, $\wedge$, F, and T, respectively, we get a new proposition p^*, which is called the *dual* of p.

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- **Duality Principle:** If two propositions (which only use the operations $'$, $\wedge$, and $\vee$) are equivalent, then their duals are equivalent. (Be lazy—save half the work!)

Example: Since the duals

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Let $k = 2n^2 + 2n \in \mathbb{Z}$. Then $m^2 = 2k + 1$, and so m^2 is odd. $\square$

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- *Proof by contradiction.* Show that if the statement to proved is false, then a contradiction results.
- *Proving the contrapositive.* Rather than directly proving an implication $p \Rightarrow q$, prove its contrapositive $q' \Rightarrow p'$.

Example: Show that if the square of an integer is even, then that integer is even.

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which we did previously. So we're done!! $\square$

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So write $\sqrt{2} = p/q$ for $p, q \in \mathbb{Z}^+$, where $q \neq 0$ and where p and q have no common factor other than 1 (i.e., the fraction p/q is "reduced to lowest terms"). Then

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$$\begin{aligned} \sqrt{2} = \frac{p}{q} \Rightarrow \frac{p^2}{q^2} = 2 \Rightarrow p^2 = 2q^2 \Rightarrow p^2 \text{ is even} \\ &\Rightarrow p \text{ is even (see previous slide)} \\ &\Rightarrow p = 2r \text{ for some positive integer } r \\ &\Rightarrow (2r)^2 = p^2 = 2q^2 \qquad \text{(Remember that } p^2 = 2q^2\text{!)} \\ &\Rightarrow 4r^2 = 2q^2 \Rightarrow 2r^2 = q^2 \Rightarrow q^2 \text{ is even} \\ &\Rightarrow q \text{ is even (again using previous slide)} \end{aligned}$$

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Solution (cont'd): We're doing a proof by contradiction. Rather than showing $\sqrt{2} \notin \mathbb{Q}$, we are trying to show how the assumption $\sqrt{2} \in \mathbb{Q}$ leads to a contradiction.

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Hence we cannot write $\sqrt{2} = p/q$ for $p, q \in \mathbb{Z}^+$, where $q \neq 0$ and where p and q have no common factor other than 1.
Hence $\sqrt{2} \notin \mathbb{Q}$. $\square$

An Example From Lewis Caroll

Given the following facts:

1. All babies are illogical.
2. Nobody is despised who can manage a crocodile.
3. Illogical persons are despised.

Prove that babies cannot manage crocodiles.

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We now have

- $b \Rightarrow d$, using (1), (3), transitive law.
- $(d')' \Rightarrow c'$, using (2), contrapositive law.

An Example From Lewis Caroll

Given the following facts:

1. All babies are illogical.
2. Nobody is despised who can manage a crocodile.
3. Illogical persons are despised.

Prove that babies cannot manage crocodiles.

Let b, c, d, and l denote the status of being a baby, being able to manage a crocodile, being despised, and being logical. Then

1. $b \Rightarrow l'$.
2. $c \Rightarrow d'$.
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We now have

- $b \Rightarrow d$, using (1), (3), transitive law.
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See text for a 10-fact example.

Predicate Logic

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Our natural conclusion? mortal(Socrates) is true.

Predicate Logic (cont'd)

- A *predicate* is a formula that contains a variable, that becomes a proposition when we substitute a particular value for the variable.
- In other words, plug in a value and get a truth value (T or F).
- Examples: $\text{man}(x)$ or $\text{mortal}(x)$.
- Can have more than one variable, e.g.,

$$\text{older}(x, y) = \text{“}x \text{ is older than } y\text{”}.$$

Predicate Logic (cont'd)

For example, suppose that four(t) means that $t \in \mathbb{Z}$ is divisible by 4 (in other words, t is an exact multiple of 4). Then:

x	four(x)	truth value of four(x)
$\vdots$	$\vdots$	$\vdots$
-4	-4 is divisible by 4	T
-3	-3 is divisible by 4	F
-2	-2 is divisible by 4	F
-1	-1 is divisible by 4	F
0	0 is divisible by 4	T
1	1 is divisible by 4	F
2	2 is divisible by 4	F
3	3 is divisible by 4	F
4	4 is divisible by 4	T
$\vdots$	$\vdots$	$\vdots$

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- Note the slight punctuation difference (comma vs. colon).

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Some Rules for Using Predicates

- **Classical syllogism:** Suppose that
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- **Negation laws:**

$$[\exists\, x \in S \colon p(x)]' \equiv [\forall\, x \in S, p'(x)]$$

and

$$[\forall\, x \in S, p(x)]' \equiv [\exists\, x \in S \colon p'(x)].$$

Predicates Having More Than One Variable

- Any given variable might not be quantified.
- The quantified variables might be quantified differently.
- **Example:** Let P be a set of people, T be a set of temperatures. Define "beach(p, t)" to mean that "person p will go to the beach if the temperature reaches t degrees".

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    Quantifying over p gives the following predicates in t:

$$\exists\, p \in P : \text{ beach}(p, t)$$
$$\forall\, p \in P, \text{beach}(p, t).$$

    Quantifying over t gives the following predicates in p:

$$\exists\, t \in T : \text{ beach}(p, t)$$
$$\forall\, t \in T, \text{beach}(p, t).$$

Predicates Having More Than One Variable (cont'd)

- Quantification example (cont'd)
  - We can quantify in both variables, getting the propositions:

$$\begin{gathered} \exists\, p \in P\colon [\exists\, t \in T\colon\ \text{beach}(p,t)] \\ \exists\, p \in P\colon [\forall\, t \in T, \text{beach}(p,t)] \\ \forall\, p \in P, [\exists\, t \in T\colon\ \text{beach}(p,t)] \\ \forall\, p \in P, [\forall\, t \in T, \text{beach}(p,t)]. \end{gathered}$$

(Many people would omit the brackets.)