CISC 1100: Structures of Computer Science

Chapter 4

Relations

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  - Problematic redundancy (joint accounts, customer with multiple accounts).
  - Break into parts to reduce redundancy:
    - Customer list: name, address, SSN, ...
    - Account list: account number, balance
    - Depositor list: account number, SSN of owner

Outline

- Ways to describe relations between objects
- Describing a relation using English
- Properties of relations

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- Use a picture.
  - Represent *domain* and *codomain* by two sets of dots.
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- Use Cartesian product of the domain and codomain, along with set builder notation to represent the relation. Sometimes we use set-based notation (e.g., "$(x, y) \in r$"), sometimes prefix notation (e.g., "$r(x, y)$") and sometimes infix notation (e.g., "$x < y$").

Describing a relation

Must specify:

- the *domain* of the relation (in language terms, the "subject" of the relation),
- the *codomain* of the relation (in language terms, the "object" of the relation), and
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Some terminology:

- When the domain and codomain are different, we have a relation *between* the two sets (or *from* the domain *to* the codomain).
- When the domain and codomain are the same, we have a relation *on* the given set.

Describing a relation (cont'd)

- **Example:** What elements are in the following relation?

Domain:	{Molly, Sandra, Mark}
Codomain:	{Molly, Sandra, Mark}
Rule:	(x, y) is in the relation iff x is sister of y.

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    $$\{(\text{Molly, Sandra}), (\text{Molly, Mark}), (\text{Sandra, Molly}), (\text{Sandra, Mark})\}$$

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- Might have two people with same hair color.
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- Might have a family member without any of given hair colors. This person would not appear in the relation.

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What elements are in the following relation?

Domain:	the set $\mathbb{N}$ of natural numbers
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- This is a relation on $\mathbb{N}$.
- r_{even} is an infinite list of ordered pairs from $\mathbb{N}$.
- Can't easily list r_{even}.
- Can *characterize* r_{even}.
  - two even numbers added will give an even number,
  - as will two odd numbers added,
  - but not an even and an odd number added.

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So r_{even} consists of pairs from $\mathbb{N}$, in which the elements of each pair are either both even or both odd.

Sometimes we use a graphical representation of a relation on a set.
Example: Consider the relation

$$\{(a, a), (a, b), (a, c), (b, b), (b, d), (e, b), (e, c), (e, d)\}$$

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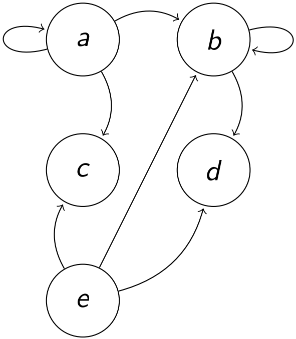
Pictorial representation:



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Tabular representation:

a	a	a	b	b	e	e	e
a	b	c	b	d	b	c	d

Properties of relations

- A relation *on a set* is one in which the domain and codomain are the same.
- A relation on a set may be any of the following:
  - reflexive
  - irreflexive
  - symmetric
  - antisymmetric
  - transitive

Reflexivity

A relation r on a set S is said to be *reflexive* if

$$(x, x) \in r \qquad \text{for any } x \in S.$$

- **Example:** The relation

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- **Example:** The relation

  Domain: $\mathbb{N}$
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  Rule: $r_{\text{odd}} = \{ (x, y) \in \mathbb{N} \times \mathbb{N} : x + y \text{ is odd} \}$.

  is not reflexive, since $(1, 1) \notin r_{\text{odd}}$.

Reflexivity and irreflexivity

- A relation r on S is said to be *irreflexive* if

$$(x, x) \notin r \qquad \text{for any } x \in S.$$

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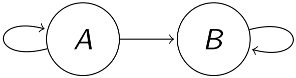
is neither reflexive nor irreflexive.

Reflexivity and irreflexivity

- The graph of a reflexive relation has a “loop” at every node.



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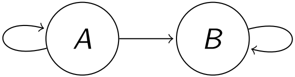

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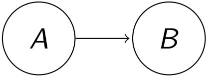

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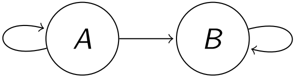

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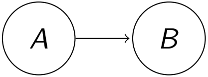
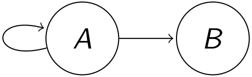


- If a relation is neither reflexive nor irreflexive, then there will be loops at some (but not all) of its nodes.



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Symmetry and antisymmetry (cont'd)

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  Example: Let $S = \{1, 2, 3\}$. The relation

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is neither symmetric nor antisymmetric.

Symmetry and antisymmetry

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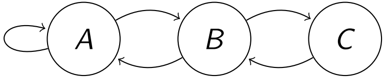

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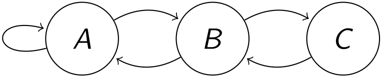

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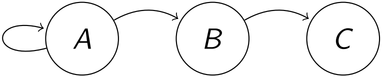

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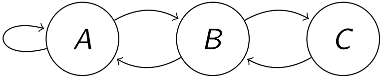

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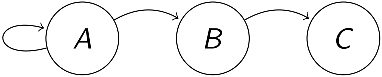
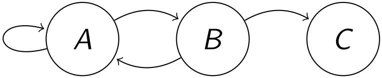


- In a relation is neither symmetric nor antisymmetric, some streets are "two-way", some are "one-way".



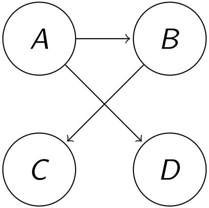
Transitivity

A relation r on a set S is said to be *transitive* if

$$(x, y) \in r \text{ and } (y, z) \in r \Rightarrow (x, z) \in r \qquad \text{for any } x, y, z \in S$$

In other words, the relation allows for shortcuts.

Transitive or intransitive?



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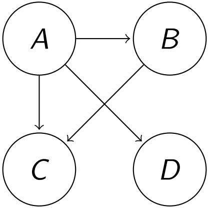

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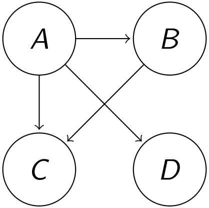

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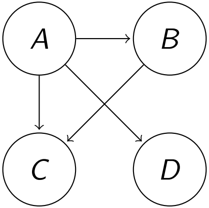

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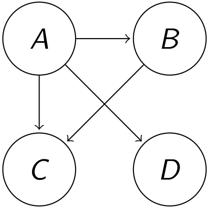

No "easy test" for transitivity:

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- Use knowledge of the relation.

Transitivity (cont'd)

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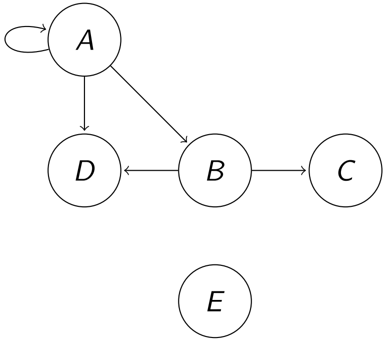

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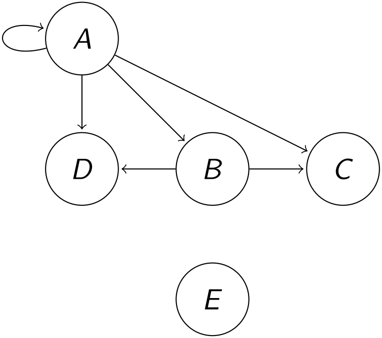

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- The "friend" relation on Facebook.

Relational Databases

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- Main ideas:
  - Store data in tables.
  - Each table has rows and columns.
  - In each table, special column called the *key*, used to identify rows. (Slight simplification.) Examples: SSN, FIDN, account number, ...
  - Key entry for each row of table must be unique.
  - Can look up row in a table by specifying its key.

Basic information for our social network is stored in the *Friends* table:

Name	City	Hometown	Sex	Birthday	Status
Alex	Topeka	Topeka	F	02/15/1996	S
Alyssa	Hartford	Albany	F	02/01/1964	M
Angela	Charlotte	Denver	F	06/15/1967	S
Anna	Hartford	Hartford	F	5/19/1989	U
Chryssi	Boston	Boston	F	12/23/1985	S
Ellen	Hartford	Boston	F	04/01/1958	M
Erik	South Park	South Park	M	08/01/1997	S
Frank	Harrisburg	Phoenix	M	12/12/1969	D
Grace	Hartford	Boston	F	02/25/1962	U
Joanna	Topeka	Topeka	F	02/15/1996	S
John	Augusta	Atlanta	M	10/25/1991	S
⋮	⋮	⋮	⋮	⋮	⋮

Relational Databases (cont'd)

The *Education* table for a social network might look like the following:

Name	University	Class	Degree	Major
Ellen	Suffolk University	1986	JD	Criminal Law
Ellen	Harvard University	1980	BA	English
Frank	Dartmouth	1996	PhD	Physics
Frank	Dartmouth	1990	BS	Physics
Grace	Fordham University	2006	MS	Computer Science
Grace	Boston College	1984	BS	Computer Science
Larry	CUNY	2007	MBA	Finance
Larry	NYU	2005	BA	Literature
Lauren	Vassar College	1985	MA	Sociology
Lauren	Duke	1983	BA	English
⋮	⋮	⋮	⋮	⋮

Relational Databases (cont'd)

Note the following:

- The *Friends* table is a 6-ary relation on

$$\text{Name} \times \text{City} \times \text{Hometown} \times \text{Sex} \times \text{Birthday} \times \text{Status}$$

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- No column of the *Education* table serves as a key.

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  - Hence Ellen was $1986 - 1958 = 28$ years old when she received her JD degree.

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- SQL, invented in the 1970s, is based on *relational algebra*, a combination of relations and logic.
- The SQL **select** operator is used to extract info from a relational database.

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and the results of the SQL **select** operation can be written as

$$\{\, (b_i, s_i) : (n_i, c_i, h_i, s_i, b_i, st_i) \in r_{\mathrm{f}} \wedge st_i = \mathrm{M} \,\}$$

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Alex	Lena
Alex	Joanna
⋮	⋮
Lena	Alex
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Since Alex is a friend of Lena and Lena is a friend of Joanna, then Alex is a FOAF of Joanna.

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This needs a little fine-tuning, to avoid the following bogus friend suggestions:

- yourself, and
- someone who's already a friend.