CISC 1100: Structures of Computer Science

Chapter 5
Functions

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Why functions?

- Sets: rigorous way to talk about collections of objects

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- Sets: rigorous way to talk about collections of objects
- Logic: rigorous way to talk about conditions and decisions
- Relations: rigorous way to talk about how objects can relate to each other
- Function: a relation in which each element of the domain is related to *exactly one* element in the codomain

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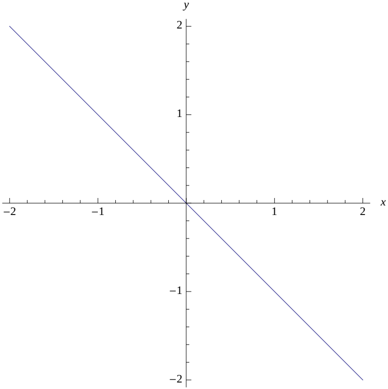
- Facebook users have email addresses, but typically only one favorite email address
  - Facebook user $\rightarrow$ email address: relation
  - Facebook user $\rightarrow$ favorite email address: function

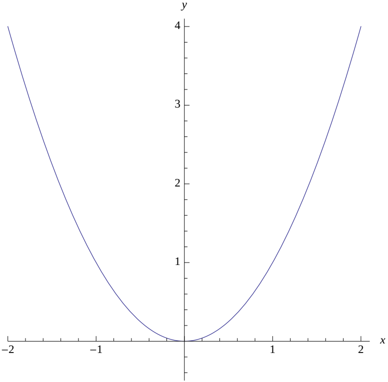
Outline

- What is a function?
- Relations and functions
- Properties of functions
- Function composition
- Identity and inverse functions
- An application: cryptography
- More about functions
- An application: secure storage of computer passwords

What is a function?

You may have already had some experience with functions, such as plotting curves such as $y = -x$ or $y = x^2$:





What is a function (cont'd)?

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- Parts of speech:
  - *domain* X: all possible inputs
  - *codomain* Y: all possible outputs
  - f: the *name* of the function (represents the rule telling assigning the output value to a given input value)
- Notation $f : X \to Y$

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- Not all functions are numerical.
- Could use English (even for numerical functions).

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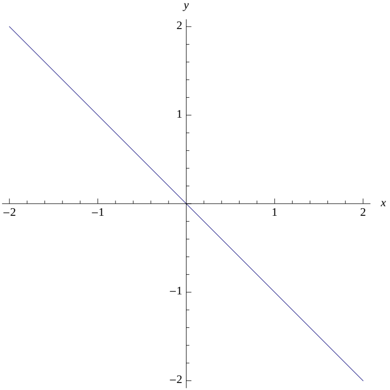
Example: For the function



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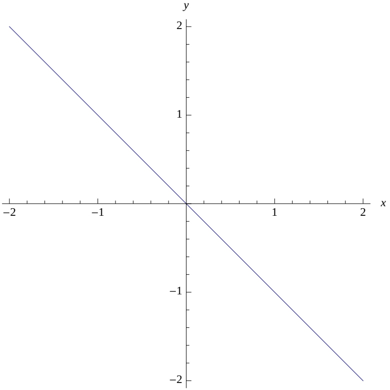

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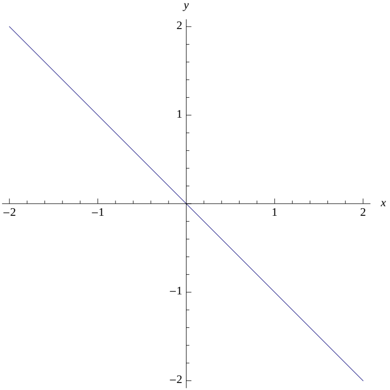

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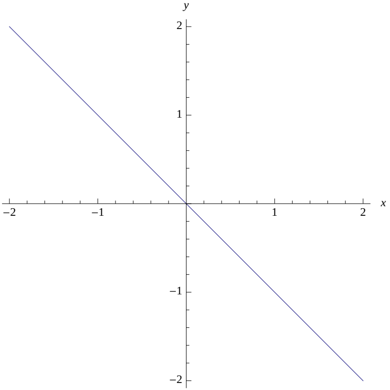

- Domain is $\mathbb{R}$.
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- Rule: This function returns the output value $-x$ for any given input value x.

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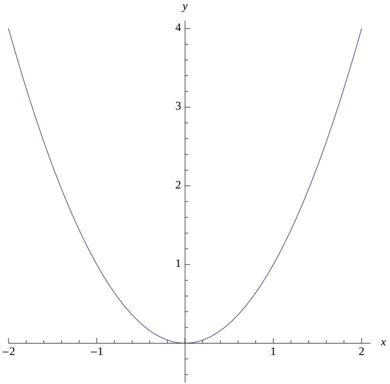

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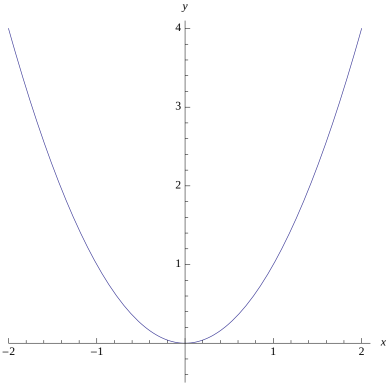

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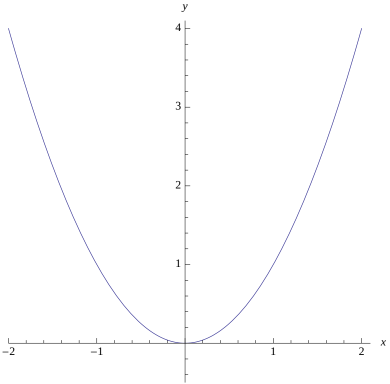

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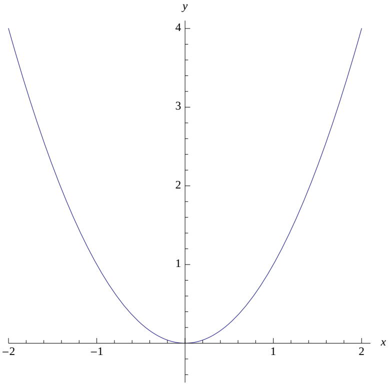

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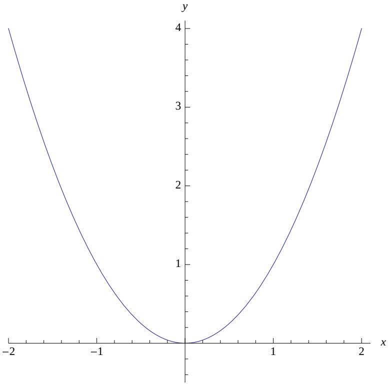

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- The functions d and d^* are different.
  Why? Different codomains!

How to describe a function (cont'd)?

Example: A function $d^{**} \colon \mathbb{Z}^+ \to \mathbb{Z}^+$ given by the table

z	1	2	3	4	5	6	7	8	9	10	...
$d^{**}(z)$	2	4	6	8	10	12	14	16	18	20	...

Example: A function $d^{**}\colon \mathbb{Z}^+ \to \mathbb{Z}^+$ given by the table

z	1	2	3	4	5	6	7	8	9	10	...
$d^{**}(z)$	2	4	6	8	10	12	14	16	18	20	...

Alternatively, can say that $d^{**}\colon \mathbb{Z}^+ \to \mathbb{Z}^+$ is given by the rule

$$d^{**}(x) = 2x \qquad \forall\, x \in \mathbb{Z}^+.$$

More examples

- Coffee shop's menu:

c	$p(c)$
small	\$1.25
medium	\$2.15
large	\$2.75

This describes a function

$$p\colon \{\text{small}, \text{medium}, \text{large}\} \to \mathbb{Q}$$

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- Bakery's menu:

i	$b(i)$
bagel	\$1.00
croissant	\$1.25
danish	\$2.25
muffin	\$1.50

  This describes a function

$$b\colon \{\text{bagel}, \text{croissant}, \text{danish}, \text{muffin}\} \to \mathbb{Q}$$

Still more examples

My address book looks something like this:

n	$e(n)$
⋮	⋮
Harry Q. Bovik	`bovik@cs.cmu.edu`
James T. Kirk	`kirk@starfleet.federation.gov`
Darth Vader	`vader@empire.gov`
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This table describes a function

$$e\colon \{\text{my friends}\} \to \{\text{all possible email addresses}\}$$

Still more examples

Each Facebook user has a gender (which s/he needn't specify):

p	$g(p)$
⋮	⋮
Bovik, Harry Q.	U
Lyons, Damian M.	M
Weiss, Gary M.	M
Papadakis-Kanaris, Christina	F
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This table describes a function

$$g\colon \{\text{all Facebook users}\} \to G$$

where $G = \{\text{M}, \text{F}, \text{U}\}$.

- If r is a relation from X to Y:
  - Some elements of X might not participate in the relation, i.e., there might be $x \in X$ such that $(x, y) \notin r$ for *any* $y \in Y$.
  - Some elements of X might be related to *more than one* element of Y, i.e., there might be $x \in X$ such that both $(x, y_1) \in r$ and $(x, y_2) \in r$, where $y_1 \neq y_2$.

Functions and relations

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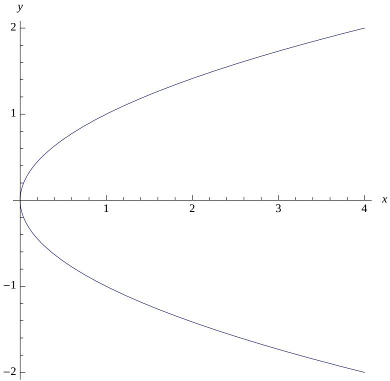
- This cannot happen with functions. If $f \colon X \to Y$, then
  - Every $x \in X$ participates in the function, i.e., $f(x)$ is defined for each $x \in X$.
  - Each $x \in X$ is associated with *exactly one* $y \in Y$, i.e., $f(x)$ is "well-defined" for each $x \in X$.

Example: The curve $x = y^2$ looks like



Does it define a function from x-values to y-values?

Functions and relations (cont'd)

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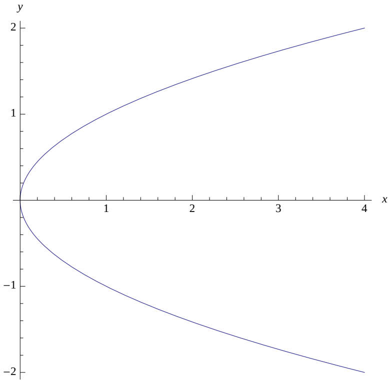

Does it define a function from x-values to y-values? No.

Let's look at some examples.

- Define $f\colon \{1,2,3,4\} \to \{1,2,3,4\}$ by

x	1	2	3
$f(x)$	3	1	2

Is f a function?

Functions and relations (cont'd)

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- Define $f: \{1, 2, 3, 4\} \to \{1, 2, 3, 4\}$ by

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- Let r be a relation from $\{1,2,3,4\}$ to $\{1,2,3,4\}$ given by

$$r = \{(1,3),(2,4),(3,1),(4,4),(1,4)\}$$

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Functions and relations (cont'd)

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- Let q be a relation from $\mathbb{R}$ to $\mathbb{R}$ defined by

$$q(x) = y \qquad \text{iff} \qquad x = y^2$$

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Moral of the story? All three pieces (the domain, the codomain, and the "rule") are important.

More terminology

- The *range* of a function is the set of all values it can assume, i.e.,
$$\text{Range}(f) = f(X) = \{\, f(x) : x \in X \,\}.$$
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  - f is *surjective* if $\forall\, t \in T, \exists\, s \in S : t = f(s)$.
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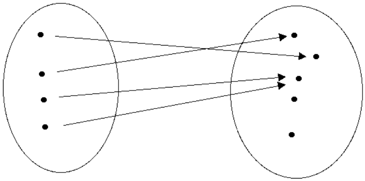
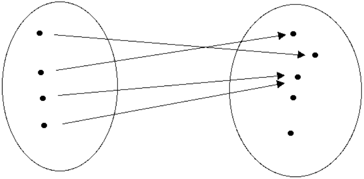
  - f is *bijective* if f is both injective and surjective.

Properties of functions (cont'd)



Not injective,
not surjective.



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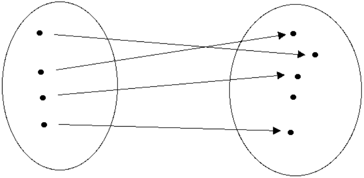

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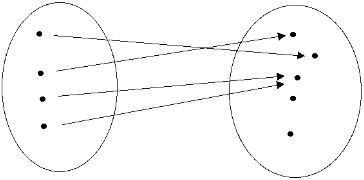
Properties of functions (cont'd)



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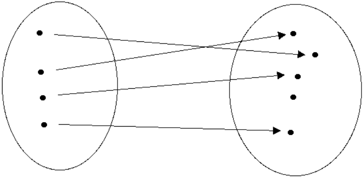

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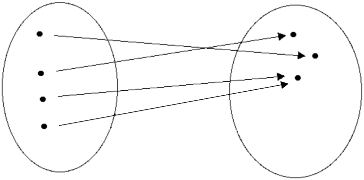

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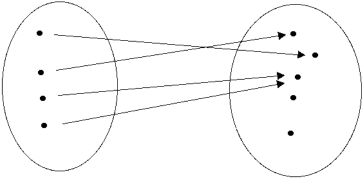

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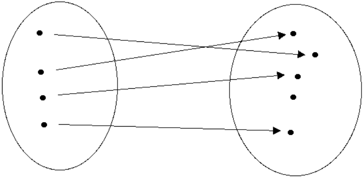

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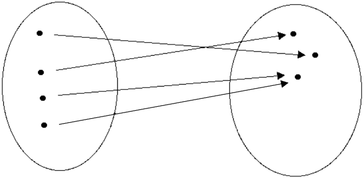

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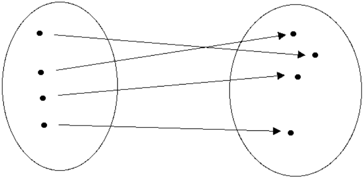

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Properties of functions (cont'd)

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    - The word "onto" is a preposition, and not an adjective.
    - Please do not say "The function f is onto."

Another way of looking at these properties:
Think of $f\colon S \to T$ as labeling S-points with T-values, i.e.,

$$s \in S \text{ is labeled by } f(s) \in T.$$

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Properties of functions (cont'd)

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Properties of functions (cont'd)

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- **Pigeonhole Principle:**

Let A and B be finite sets.

1. If $|A| < |B|$, then there can be no surjection from A to B.
2. If $|A| > |B|$, then there can be no injection from A to B.
3. If $|A| \neq |B|$, then there can be no bijection from A to B.

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- Thus, there exist distinct i and j such that $p(i) = p(j)$.
- So players i and j must be on the same team. □

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- All this is part of *software engineering*.

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  Then
$$h(x) = g\big(f(x)\big) \qquad \forall\, x \in \mathbb{R}.$$

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- Let $f\colon X \to Y$ and $g\colon Y \to Z$. The *composite function* $g \circ f\colon X \to Z$ is defined as

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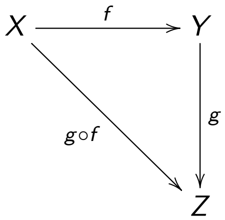

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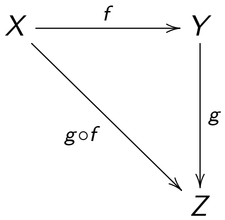

- Although we write $g \circ f$ and we read g before f when we say "g composed with f," we first calculate $y = f(x)$ and then $z = g(y)$ when we compute $z = g\big(f(x)\big)$.

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However $d \circ p\colon \mathbb{Z} \to \mathbb{R}$ *is* well-defined.

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  We have

$$(f \circ g)(2) = f(g(2)) = f(2+1) = f(3) = 2 \times 3 = 6;$$

Function composition (cont'd)

- Let $f\colon X \to Y$ and $g\colon Y \to Z$. Computing the composite function $g \circ f$ at a point $x \in X$ is a two-step process:
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  Then $z = (g \circ f)(x)$.
- **Example:** Define $f, g\colon \mathbb{R} \to \mathbb{R}$ by

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Function composition is not commutative!

Function composition (cont'd)

Example: Let P be the set of all people. Define functions $f\colon P \to P$ and $m\colon P \to P$ by

$$f(p) = \text{the (birth) father of } p \qquad \forall\, p \in P$$

and

$$m(p) = \text{the (birth) mother of } p \qquad \forall\, p \in P.$$

What is $m \circ m$? $m \circ f$? $f \circ m$? $f \circ f$?

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Solution: We have $(m \circ m)(p) = m\bigl(m(p)\bigr)$, which is the mother of the mother of p, i.e., the maternal grandmother of p. Similarly, we find that

$$\begin{aligned} m \circ f &= \text{paternal grandmother}, \\ f \circ m &= \text{maternal grandfather}, \\ f \circ f &= \text{paternal grandfather}. \end{aligned}$$

Identity and inverse functions

- The *identity function* on a set A is the function $\mathrm{id}_A \colon A \to A$ defined by

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- Why this name? Analogous to

$$a \times 1 = a = 1 \times a \qquad \forall\, a \in \mathbb{R}.$$

- **Example:** Let V be the set of all vowels. The identity function $\mathrm{id}_V \colon V \to V$ is given by

x	a	e	i	o	u
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- **Example:** Let C be the set of all consonants. The identity function $\mathrm{id}_C \colon C \to C$ is similar:

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- Why do we care about a function that "does nothing"?

Identity and inverse functions (cont'd)

- The function $f\colon X \to Y$ is *invertible* if there exists another function $f^{-1}\colon Y \to X$ such that

$$f^{-1} \circ f = \mathrm{id}_X \qquad \text{and} \qquad f \circ f^{-1} = \mathrm{id}_Y,$$

i.e.,

$$\begin{aligned} f^{-1}\bigl(f(x)\bigr) &= x \qquad \forall\, x \in X \\ f\bigl(f^{-1}(y)\bigr) &= y \qquad \forall\, y \in Y. \end{aligned}$$

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- Don't confuse f^{-1} with a reciprocal $(1/f)$!

Identity and inverse functions (cont'd)

Example: Define $g\colon \mathbb{Z} \to \mathbb{Z}$ by

$$g(x) = x - 7 \qquad \forall\, x \in \mathbb{Z}.$$

Show that $g^{-1}\colon \mathbb{Z} \to \mathbb{Z}$ is given by

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$$\begin{aligned}(g \circ g^{-1})(y) &= g\big(g^{-1}(y)\big) = g(y+7) \\ &= (y+7) - 7 = y\end{aligned} \qquad \forall\, y \in \mathbb{Z}$$

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So g^{-1} is the functional inverse of g, as claimed.

Identity and inverse functions (cont'd)

Not all functions are invertible, and the difference between invertibility and non-invertibility may be subtle.

- **Example:** Define $m\colon \mathbb{Q} \to \mathbb{Q}$ by

$$m(x) = 2x \qquad \forall\, x \in \mathbb{Q}.$$

  Is m invertible? If so, what is its inverse function?

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  for x in terms of y:

$$y = 2x \iff x = \tfrac{1}{2}y.$$

  Now $y \in \mathbb{Q} \implies x = \frac{1}{2}y \in \mathbb{Q}$. Thus m is invertible, with $m^{-1}\colon \mathbb{Q} \to \mathbb{Q}$ given by

$$m^{-1}(y) = \tfrac{1}{2}y \qquad \forall\, y \in \mathbb{Q}. \quad \square$$

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- **Example:** Define $\widetilde{m}\colon \mathbb{N} \to \mathbb{N}$ by

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Is $\widetilde{m}$ invertible? If so, give its inverse function.

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  No! For example, take $y = 1$, getting $x = \frac{1}{2}$.
  So $\widetilde{m}$ is *not* invertible.

Identity and inverse functions (cont'd)

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- For any $y \in Y$, there must *exist* a *unique* $x \in X$ such that $x = f^{-1}(y)$, i.e., such that $y = f(x)$.
  - *Uniqueness* holds iff f is an injection.

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    If $y = f(x)$ and also $y = f(x')$, we wouldn't know whether we should use x or x' as the value of $f^{-1}(y)$.

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  - *Uniqueness* holds iff f is an injection.
    If $y = f(x)$ and also $y = f(x')$, we wouldn't know whether we should use x or x' as the value of $f^{-1}(y)$.
  - *Existence* holds iff for any $y \in Y$, there exists some $x \in X$ such that $f(x) = y$, i.e., iff f is a surjection.

Identity and inverse functions (cont'd)

Which of the following functions are invertible?

- A function from the set $\{1, 2, 3, \ldots, 999\}$ to the set $\{1, 2, 3, \ldots, 999, 1000\}$.

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- The function $q\colon \{1, 2, 3, 4\} \to \{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\}$ defined by

τ	1	2	3	4
$q(\tau)$	$\spadesuit$	$\heartsuit$	$\diamondsuit$	$\clubsuit$

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$q(\tau)$	$\spadesuit$	$\heartsuit$	$\diamondsuit$	$\clubsuit$

  Yes. q is a bijection. In fact its inverse is the function $q^{-1}\colon \{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\} \to \{1, 2, 3, 4\}$ defined by

s	$\clubsuit$	$\diamondsuit$	$\heartsuit$	$\spadesuit$
$q^{-1}(s)$	4	3	2	1

Identity and inverse functions (cont'd)

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- The function $f\colon \mathbb{R} \to \mathbb{R}^{\geq 0}$ defined by

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- The function $f\colon \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ defined by

  $$f(x) = x^2 \qquad \forall\, x \in \mathbb{R}.$$

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- The function $f\colon \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ defined by

  $$f(x) = x^2 \qquad \forall\, x \in \mathbb{R}.$$

  Yes! Its inverse is the function $f^{-1}\colon \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ defined by

  $$f^{-1}(y) = \sqrt{y} \qquad \forall\, y \in \mathbb{R}^{\geq 0}.$$

Identity and inverse functions (cont'd)

One last example: Let $g\colon \mathbb{R} \to \mathbb{R}$ be defined by

$$g(s) = 4s - 3 \qquad \forall\, s \in \mathbb{R}\,.$$

Let's find an explicit formula for $g^{-1}\colon \mathbb{R} \to \mathbb{R}$.

- By definition, we know that $s = g^{-1}(t)$ is equivalent to $t = g(s)$.

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- Using simple algebra, we have

  $$t = 4s - 3$$

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  for s in terms of t, then $s = g^{-1}(t)$.
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Identity and inverse functions (cont'd)

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Let's find an explicit formula for $g^{-1}\colon \mathbb{R} \to \mathbb{R}$.

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- Thus $g^{-1}\colon \mathbb{R} \to \mathbb{R}$ is given by the rule

  $$g^{-1}(t) = \tfrac{1}{4}(t+3) \qquad \forall\, t \in \mathbb{R}\,.$$

Identity and inverse functions (cont'd)

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This expression on the right-hand side is precisely $f^{-1}(y)$.

Inverse of composite functions

Fact: Let $f\colon A \to B$ and $g\colon B \to C$ be invertible functions. Then $g \circ f\colon A \to C$ is invertible, with

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Inverse of composite functions (cont'd)

Example: Define $f, g\colon \mathbb{R} \to \mathbb{R}$ by

$$f(x) = 2x + 7 \text{ and } g(x) = x^3 - 8 \qquad \forall\, x \in \mathbb{R}.$$

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- This extends to compositions of any number of functions, e.g.,

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- Useful in Alice, Part III (unmelting the snow woman).

An example: cryptography

Consider the following scenarios:

- When you purchase an item from an e-business, you submit (among other things) a credit card number.
  If this information is intercepted when it is transmitted to the online store, you are a prime candidate for identity theft.

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These are problems in computational cryptography, which deals with the problem of hiding information from people who shouldn't see it.

An example: cryptography (cont'd)

Julius Caesar needed to securely send military messages to his troops. Given the original *cleartext*, he created a *ciphertext* by replacing each letter by the one that comes three positions later in alphabetical order (*Caesar rotation*). This defines a encoding function $e\colon \{\mathtt{A}, \mathtt{B}, \dots, \mathtt{Z}\} \to \{\mathtt{A}, \mathtt{B}, \dots, \mathtt{Z}\}$, defined by the table

x	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
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Corresponding decoding function $d\colon \{\mathtt{A}, \mathtt{B}, \ldots, \mathtt{Z}\} \to \{\mathtt{A}, \mathtt{B}, \ldots, \mathtt{Z}\}$ is the inverse of the encoding function:

$$e\bigl(d(x)\bigr) = x \qquad \text{and} \qquad d\bigl(e(x)\bigr) = x$$

for any $x \in \{\mathtt{A}, \mathtt{B}, \ldots, \mathtt{Z}\}$. More succinctly,

$$e \circ d = \mathrm{id}_{\{\mathtt{A},\mathtt{B},\ldots,\mathtt{Z}\}} = d \circ e.$$

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- If an enemy were to see this message and if he didn't know the secret, he'd simply dismiss it as gibberish.

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For Caesar rotation $e\colon \{\mathtt{A},\mathtt{B},\ldots,\mathtt{Z}\} \to \{\mathtt{A},\mathtt{B},\ldots,\mathtt{Z}\}$

x	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
$e(x)$	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C

decoding function $d\colon \{\mathtt{A},\mathtt{B},\ldots,\mathtt{Z}\} \to \{\mathtt{A},\mathtt{B},\ldots,\mathtt{Z}\}$ is

y	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
$d(y)$	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W

("swap the rows").

For example, `ATTACK AT NOON` encodes as `DWWDFN DW QRRQ`.

- If an enemy were to see this message and if he didn't know the secret, he'd simply dismiss it as gibberish.
- But Caesar's forces (who had already been told what the encoding and decoding methods were), *would* be able to decode it!

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- Can we do this?

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  No "security through obscurity"!
- This does *not* mean that these techniques are provably secure!
  - Nobody knows how to do fast factorization.
  - Nobody has ever proved that fast factorization is impossible!

More about functions

Where else do functions crop up in computer science?

Standard mathematical functions Here's a partial list of functions you may have encountered:

math name	UNIX name	description
$\sqrt{\ }$	`sqrt`	square root
$\sin$	`sin`	trigonometric sine
$\cos$	`cos`	trigonometric cosine
$\tan$	`tan`	trigonometric tangent
$\sin^{-1}$	`asin`	trigonometric arc (inverse) sine
$\cos^{-1}$	`acos`	trigonometric arc cosine
$\tan^{-1}$	`atan`	trigonometric arc tangent
$\exp$	`exp`	exponential function
$\ln$	`log`	natural logarithm
$\lvert \cdot \rvert$	`fabs`	absolute value

More about functions (cont'd)

Standard mathematical functions (cont'd) You may be less familiar with the following:

- The max function. If x and y are numbers, then $\max(x, y)$ is the maximum of x and y. For example, $\max(2.3, -4.2) = 2.3$.

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- The ceiling function. If x is a number, then $\lceil x \rceil$ is the smallest integer that is greater than or equal to x. For example, $\lceil 4.001 \rceil = 5$.

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The names of these functions, as found in the UNIX standard library, are `fmax`, `fmin`, `ceil`, and `floor`.

More about functions (cont'd)

Growth functions: Used to measure efficiency of algorithms.
Typically a function $f : \mathbb{Z} \to \mathbb{Z}$, with

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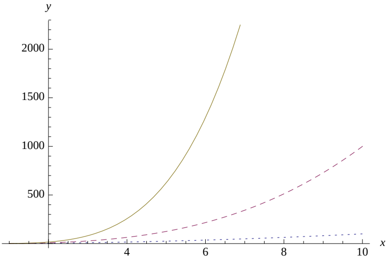
Here are some standard growth functions:

function	name
$\log n$	logarithmic
n	linear
$n \log n$	(no commonly-accepted name)
n^2	quadratic
n^3	cubic
2^n	exponential
$n!$	factorial

Growth functions (cont'd):

Let's do some graphing.



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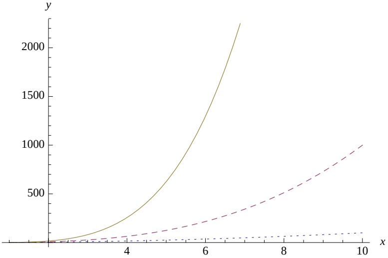

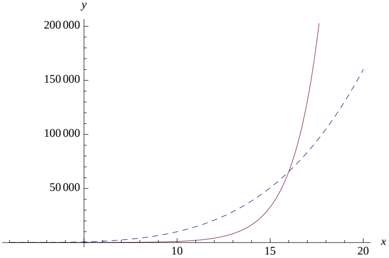

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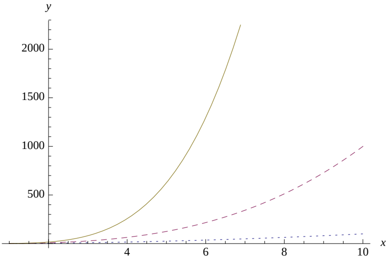

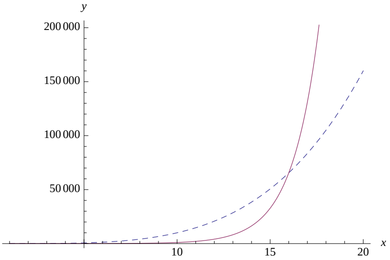

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Breakpoint (between tractable and intractable problems): polynomial vs. exponential

More about functions (cont'd)

Functions in program construction
Functions are ubiquitous in the design and implementation of computer programs. For starters, functions are the main building block for many computer programming languages. For instance, every executable C or C++ program will have a function named `main`, which is the starting point for program execution.

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Functions are ubiquitous in the design and implementation of computer programs. For starters, functions are the main building block for many computer programming languages. For instance, every executable C or C++ program will have a function named `main`, which is the starting point for program execution.

An example in C++:

```
#include <iostream>

int main()
{
   std::cout << "Hello, world!\n";
}
```

Functions in program construction (cont'd):

Look at the following:

```
int main()
{
   do_initialization();
   do {
      data = get_input_data();
      result = process_data(data);
      put_result(result);
      still_working = more_to_process();
   } while (still_working);
   do_cleanup();
}
```

More about functions (cont'd)

Functions in program construction (cont'd): This particular `main` function involves other functions. Note the following points:

- This is a syntactically correct C++ (or C) `main` function.
- This could be the `main` function for almost *any* text-based task.
- `main` involves other functions. These can be written by other programmers. In fact, they themselves can involve (sub)functions, and so on. Can use this "functional decomposition" to split the work amongst a team of programmers.
- At each stage, we have a working system (without all the features).
- When functions are fully fleshed out, we have a complete working system.
- This approach can make testing a lot easier.

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- System stores each user's encrypted password in a world-readable file.

An application: secure storage of passwords (cont'd)

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- User is allowed in if and only if $\tilde{e} = e$.

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- If f^{-1} can be computed quickly, then a Bad Guy could compute plaintext password, given the encrypted password.

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The moral of the story: choose good passwords!