### CISC 1100: Structures of Computer Science Chapter 7 Probability

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Historical note: A gambler's dispute in 1654 led Blaise Pascal and Pierre de Fermat to create a mathematical theory of probability.

- Terminology and background
- Complement
- Elementary rules for probability
- General rules for probability
- Bernoulli trials and probability distributions
- Expected value

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- Event: set E of desired outcomes
- Sample space: (finite) set S of all possible outcomes
- The probability Prob(E) of an event  $E \subseteq S$  is given as

$$\mathsf{Prob}(E) = \frac{|E|}{|S|}$$

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### Solution:

• Our sample space *S* is

$$S = \{(1, 1), (1, 2), \dots, (1, 6),$$
  
 $(2, 1), (2, 2), \dots, (2, 6),$   
 $(3, 1), (3, 2), \dots, (3, 6),$   
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So

$$Prob(E) = \frac{|E|}{|S|} = \frac{6}{36} = \frac{1}{6} = 0.1667.$$

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- If either *E* or *S* is big, this is impractical. (Poker problems, lottery problems, ...).
- Can often use counting principles from previous chapter to determine |S| and/or |E|.
- In our case, there are 6 outcomes for the roll of each of the two dice.
- So multiplication principle tells us that there are 6 × 6 = 36 outcomes for the roll of both dice, i.e., |S| = 36.

• Since  $0 \le |E| \le |S|$  and  $\operatorname{Prob}(E) = |E|/|S|$ , we have

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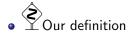
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• If you ever calculate a probability as being negative or being greater than 1, you've made a mistake.



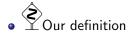
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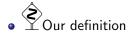
relies on two assumptions:

• |S| is finite.



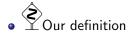
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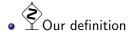
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- |S| is finite.
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- If these don't hold, the formula is incorrect.
  - Example: Throwing a loaded die.
  - Example: Choosing a ball out of a bag, where some balls are larger than others.

$$S = \{(H,H,H), (H,H,T), (H,T,H), (H,T,T), (T,H,H), (T,H,T), (T,T,H), (T,T,T)\}$$

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$$Prob(E) = \frac{|E|}{|S|} = \frac{1}{8} = 0.125.$$

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• Of course, we really know |E| = 1 directly, since

$$E = \{(H, H, H, H, H, H, H, H, H, H)\}$$

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$$\begin{split} S &= \{ (\mathsf{H},\mathsf{H},\mathsf{H}), \ (\mathsf{H},\mathsf{H},\mathsf{T}), \ (\mathsf{H},\mathsf{T},\mathsf{H}), \ (\mathsf{H},\mathsf{T},\mathsf{T}), \ (\mathsf{T},\mathsf{H},\mathsf{H}), \ (\mathsf{T},\mathsf{H},\mathsf{T}), \\ &\quad (\mathsf{T},\mathsf{T},\mathsf{H}), \ (\mathsf{T},\mathsf{T},\mathsf{T}) \} \\ E &= \{ (\mathsf{H},\mathsf{H},\mathsf{T}), \ (\mathsf{H},\mathsf{T},\mathsf{H}), \ (\mathsf{T},\mathsf{H},\mathsf{H}) \} \\ \mathsf{Prob}(E) &= \frac{|E|}{|S|} = \frac{3}{8} = 0.375 \, . \end{split}$$

This is an example of a Bernoulli trial. We'll look at this later.

Solution:

|S| = 52,

$$\begin{split} |S| &= 52, \\ E &= \{2\clubsuit, 2\diamondsuit, 2\heartsuit, 2\clubsuit, 3\clubsuit, 3\diamondsuit, 3\heartsuit, 3\diamondsuit\}, \text{ so that } |E| = 8, \end{split}$$

$$|S| = 52,$$
  
 $E = \{2, 2, 2, 2, 2, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 4\},$  so that  $|E| = 8,$   
 $Prob(E) = \frac{8}{52} = \frac{2}{13} = 0.154.$ 

#### Complement

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• Probability of a complement?

$$\operatorname{Prob}(E') = 1 - \operatorname{Prob}(E).$$

• Why? Since

$$|E'|=|S|-|E|,$$

it follows that

$$Prob(E') = \frac{|E'|}{|S|} = \frac{|S| - |E|}{|S|} = \frac{|S|}{|S|} - \frac{|E|}{|S|} = 1 - Prob(E).$$

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  - computing Prob(E) = 1 Prob(E').

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- Easy to see that  $|S| = 12^9 = 515,978,035,235$ .
- Calculating |E|: seems hard, so calculate |E'| instead:

$$|E'| = P(12,9) = 12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 = 79,833,600.$$

Thus

$$\mathsf{Prob}(E') = \frac{|E'|}{|S|} = \frac{79,833,600}{515,978,035,235} = 0.01547 = 15.47\%$$

So

$$Prob(E) = 1 - Prob(E') = 1 - 0.01547 = 0.98453 = 98.453\%$$

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- Similarly, there are addition and multiplication rules for probability.
- To properly state these rules, we need two new concepts: *disjointness* and *independence*:
  - Two or more events are *disjoint* if the outcomes associated with one event are not present in the outcomes of any of the other events (i.e., if the events form non-overlapping sets). More formally, two events E<sub>1</sub> and E<sub>2</sub> are *disjoint* if E<sub>1</sub> ∩ E<sub>2</sub> = Ø.

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  - Two events  $E_1$  and  $E_2$  are *independent* if the outcome of any one of these events does not *in any way* impact or influence the outcome of the other event.

More formally, two events  $E_1$  and  $E_2$  are *independent* if

 $\operatorname{Prob}(E_1 \cap E_2) = \operatorname{Prob}(E_1) \cdot \operatorname{Prob}(E_2).$ 

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- Solution: Let

$$E_1 =$$
 "roll an odd number" =  $\{1, 3, 5\}$ 

and

$$E_2 =$$
 "roll an even number" = {2, 4, 6}.

Since  $E_1 \cap E_2 = \emptyset$  contains no elements, the events are disjoint.

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- Solution: The events are not independent.
  - Using the informal criterion: If we know that event  $E_1$  has occurred, then event  $E_2$  can *never* occur. Moreover, if  $E_1$  has not occurred, then  $E_2$  *must* occur. Since the outcome of  $E_1$  influences the outcome of  $E_2$ , the events  $E_1$  and  $E_2$  are not independent.

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    - We know that  $Prob(E_1) = \frac{3}{6} = \frac{1}{2}$  and  $Prob(E_2) = \frac{3}{6} = \frac{1}{2}$ . Hence  $Prob(E_1) \cdot Prob(E_2) = \frac{1}{4}$ .

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    - Since  $E_1 \cap E_2 = \emptyset$ , we find that  $Prob(E_1 \cap E_2) = 0$ .
    - So  $\operatorname{Prob}(E_1) \cdot \operatorname{Prob}(E_2) \neq \operatorname{Prob}(E_1 \cap E_2)$ .

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  - Using the informal criterion: If we know that event  $E_1$  has occurred, then event  $E_2$  can *never* occur. Moreover, if  $E_1$  has not occurred, then  $E_2$  *must* occur. Since the outcome of  $E_1$  influences the outcome of  $E_2$ , the events  $E_1$  and  $E_2$  are not independent.
  - Using the formal criterion:
    - We know that  $Prob(E_1) = \frac{3}{6} = \frac{1}{2}$  and  $Prob(E_2) = \frac{3}{6} = \frac{1}{2}$ . Hence  $Prob(E_1) \cdot Prob(E_2) = \frac{1}{4}$ .
    - Since  $E_1 \cap E_2 = \emptyset$ , we find that  $Prob(E_1 \cap E_2) = 0$ .
    - So  $\operatorname{Prob}(E_1) \cdot \operatorname{Prob}(E_2) \neq \operatorname{Prob}(E_1 \cap E_2)$ .

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Second part establishes a useful general result: *disjoint events* (having non-zero probabilities) are never independent.

• **Example:** Two cards are drawn from a fresh deck without replacement. Are the outcomes associated with each draw disjoint?

- **Example:** Two cards are drawn from a fresh deck without replacement. Are the outcomes associated with each draw disjoint?
- **Solution:** Let  $E_1$  and  $E_2$  denote the events of drawing the first and second cards from the deck. Clearly  $E_1 \cap E_2$  consists of 51 possibilities. Thus  $E_1 \cap E_2 \neq \emptyset$ , and so the outcomes are *not* disjoint.

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- For example, if A is drawn on the first draw then it cannot be drawn on the second draw.

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Formal criterion:

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- Clearly,  $\operatorname{Prob}(E_1) = \frac{1}{52}$ .
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- Clearly,  $\operatorname{Prob}(E_1) = \frac{1}{52}$ .
- With no knowledge of  $E_2$ , we have  $Prob(E_2) = \frac{1}{52}$ .
- So  $Prob(E_1) \cdot Prob(E_2) = \frac{1}{52} \cdot \frac{1}{52} = \frac{1}{2,704}$ .

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- So  $Prob(E_1) \cdot Prob(E_2) = \frac{1}{52} \cdot \frac{1}{52} = \frac{1}{2.704}$ .
- Since Prob(E<sub>1</sub> ∩ E<sub>2</sub>) ≠ Prob(E<sub>1</sub>) · Prob(E<sub>2</sub>), the events are not independent.

• Let  $E_1$  and  $E_2$  be disjoint events. Then

$$\mathsf{Prob}(E_1 \cup E_2) = \mathsf{Prob}(E_1) + \mathsf{Prob}(E_2).$$

## Addition rule for probability

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• Why?

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• Since  $E_1$  and  $E_2$  are disjoint,

 $|E_1 \cup E_2| = |E_1| + |E_2|$ 

(recall from chapter on sets).

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• Why?

• Since  $E_1$  and  $E_2$  are disjoint,

$$|E_1 \cup E_2| = |E_1| + |E_2|$$

(recall from chapter on sets). • So  $\operatorname{Prob}(E_1 \cup E_2) = \frac{|E_1 \cup E_2|}{|S|} = \frac{|E_1|}{|S|} + \frac{|E_2|}{|S|}$   $= \operatorname{Prob}(E_1) + \operatorname{Prob}(E_2).$ 

• **Example:** What is the probability of drawing a blackjack in the initial deal from a fresh deck?

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We have

 $\mathsf{Prob}(\mathsf{ace}, \mathsf{then value}{=}10) = \mathsf{Prob}(\mathsf{ace on first draw})$ 

 $\times$  Prob(value=10 on second draw)

$$=rac{4}{52} imesrac{16}{51}=0.024$$

Prob(value=10, then ace) = Prob(value=10 on first draw)

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• Thus Prob(blackjack) = 0.024 + 0.024 = 0.048 (or 4.8%).

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- **Example:** A person rolls a die and wins a prize if the roll is a 1 or a 2. What is the probability of winning?
- Solution: The events  $E_1 = \{1\}$  and  $E_2 = \{2\}$  are disjoint. So

$$\begin{aligned} \mathsf{Prob}(1 \text{ or } 2) &= \mathsf{Prob}(E_1 \cup E_2) = \mathsf{Prob}(E_1) + \mathsf{Prob}(E_2) \\ &= \frac{1}{6} + \frac{1}{6} = \frac{1}{3} = 0.333 \,. \end{aligned}$$

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• This was easy enough to do directly, since we're interested in Prob(E) for  $E = \{1, 2\}$ . So

$$\operatorname{Prob}(E) = \frac{|E|}{|S|} = \frac{2}{6} = \frac{1}{3} = 0.333.$$

#### • If $E_1$ and $E_2$ are independent events, then

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#### • Why? This is simply the formal definition of independence.

• **Example:** Determine the probability that you flip a coin three times and that you get exactly three heads.

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#### Constituent events

$$E_i =$$
 "toss # i is a head"  $(i = 1, 2, 3)$ 

are intuitively independent, and so

$$Prob(E) = Prob(E_1 \cap E_2 \cap E_3)$$
  
= Prob(E\_1) \cdot Prob(E\_2) \cdot Prob(E\_3).

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• Clearly

$$\mathsf{Prob}(E_1) = \mathsf{Prob}(E_2) = \mathsf{Prob}(E_3) = \frac{1}{2}.$$

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Clearly

$$\mathsf{Prob}(E_1) = \mathsf{Prob}(E_2) = \mathsf{Prob}(E_3) = \frac{1}{2}.$$

So

$$Prob(E) = \frac{1}{8} = 0.125.$$

• color (8 choices, including red)

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- A/C (yes or no)

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Can't decide, so choose randomly. What is the probability that you will wind up with a red car, with air-conditioning, but without the 4-wheel drive option?

• **Solution:** Let *E* denote the event of interest (red car, A/C, no 4WD). Constituent events are

 $E_1 =$  "choose red color",

 $E_2 =$  "choose A/C option",

 $E_3 =$  "don't choose 4WD option".

with corresponding sample spaces of sizes

$$|S_1| = 8, |S_2| = 2, |S_3| = 2.$$

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$$|S_1| = 8, |S_2| = 2, |S_3| = 2.$$

• Since  $|E_1| = |E_2| = |E_3| = 1$ , we have  $Prob(E_1) = \frac{1}{8}$ ,  $Prob(E_2) = \frac{1}{2}$ ,  $Prob(E_3) = \frac{1}{2}$ .

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• Since the events are intuitively independent, we have

$$\begin{aligned} \mathsf{Prob}(E) &= \mathsf{Prob}(E_1 \cap E_2 \cap E_3) \\ &= \mathsf{Prob}(E_1) \cdot \mathsf{Prob}(E_2) \cdot \mathsf{Prob}(E_3) \\ &= \frac{1}{8} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{32} = 0.031 \,. \end{aligned}$$

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  - Note that  $|E_1 \cup E_2| \neq |E_1| + |E_2| = 30$ . Since there are two black 2's and two red 2's, we don't want to double-count the red 2's! So  $|E_1 \cup E_2| = 28$ .

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  - So

$$\mathsf{Prob}(E_1 \cup E_2) = rac{|E_1 \cup E_2|}{|S|} = rac{28}{52} = 0.538$$

### General addition rule for probability

• For any two events  $E_1$  and  $E_2$ ,

 $\mathsf{Prob}(E_1 \cup E_2) = \mathsf{Prob}(E_1) + \mathsf{Prob}(E_2) - \mathsf{Prob}(E_1 \cap E_2)$ 

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• Why? Simple consequence of the inclusion/exclusion rule

$$|E_1 \cup E_2| = |E_1| + |E_2| - |E_1 \cap E_2|$$

• Note that this reduces to the earlier rule

 $Prob(E_1 \cup E_2) = Prob(E_1) + Prob(E_2)$  for disjoint  $E_1, E_2$ 

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  - Let  $E_2 =$  "pick a 2". Then  $|E_2| = 4$ , and so  $Prob(E_2) = \frac{4}{52} = \frac{1}{13}$ .

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  - Since there are two red 2's,  $|E_1 \cap E_2| = 2$ , and so  $Prob(E_1 \cap E_2) = \frac{2}{52} = \frac{1}{26}$ .

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  - |S| = 52, as before.
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  - Let  $E_2 =$  "pick a 2". Then  $|E_2| = 4$ , and so  $Prob(E_2) = \frac{4}{52} = \frac{1}{13}$ .
  - Since there are two red 2's,  $|E_1 \cap E_2| = 2$ , and so  $Prob(E_1 \cap E_2) = \frac{2}{52} = \frac{1}{26}$ .
  - Thus

$$Prob(E_1 \cup E_2) = Prob(E_1) + Prob(E_2) - Prob(E_1 \cap E_2)$$
$$= \frac{1}{2} + \frac{1}{13} - \frac{1}{26} = \frac{7}{13}$$
$$= 0.538$$

• **Example:** I flip a coin and roll a six-sided die. What is the probability that I get a head or roll either a 1 or a 2 ?

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- Solution:
  - We have

$$\begin{split} S_1 &= \{\text{head, tails}\}, \text{ and so } |S_1| = 2, \\ S_2 &= \{1, 2, 3, 4, 5, 6\}, \text{ and so } |S_2| = 6, \\ E_1 &= \text{``flip coin and get head'', and so } |E_1| = 1, \\ E_2 &= \text{``roll die and get 1 or 2'', and so } |E_2| = 2. \end{split}$$

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Thus

$$Prob(E_1) = \frac{|E_1|}{|S_1|} = \frac{1}{2}$$
$$Prob(E_2) = \frac{|E_2|}{|S_2|} = \frac{2}{6} = \frac{1}{3}.$$

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Hence

$$\begin{aligned} \mathsf{Prob}(E_1 \cup E_2) &= \mathsf{Prob}(E_1) + \mathsf{Prob}(E_2) - \mathsf{Prob}(E_1 \cap E_2) \\ &= \frac{1}{2} + \frac{1}{3} - \frac{1}{6} = \frac{2}{3} = 0.667 \,. \end{aligned}$$

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- If  $E_1$ ,  $E_2$  are independent, then  $Prob(E_1|E_2) = Prob(E_1)$ .

• Example: A standard six-sided die is rolled by your friend, but in such a way that you cannot see what value comes up. Your friend tells you that the value that comes up is odd, but does not tell you what specific value was rolled. What is the probability that she rolled a 3?

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- Since the roll was odd,  $E_2 = \{1, 3, 5\}$ .
- Since  $|E_2| = 3$  and one of the three elements of  $E_2$  is 3,

$$\mathsf{Prob}(E_1|E_2) = \frac{1}{3}.$$

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- But these  $E_1$  elements must really be elements of  $E_1 \cap E_2$ .

• The general multiplication rule is now given by

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Second line:

$$Prob(E_1 \cap E_2) = Prob(E_2 \cap E_1)$$
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  - $\operatorname{Prob}(E_2|E_1) = \frac{3}{51}$ .
  - So

$$\begin{aligned} \mathsf{Prob}(E_1 \cap E_2) &= \mathsf{Prob}(E_1) \cdot \mathsf{Prob}(E_2 | E_1) \\ &= \frac{4}{52} \times \frac{3}{51} = 0.0045 \,. \end{aligned}$$

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• So the probability is given by

$$C(10,2) \cdot (\frac{1}{2})^2 \cdot (\frac{1}{2})^8 = 45 \cdot (\frac{1}{2})^{10} = \frac{45}{1024} = 0.0439$$
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• So the probability is given by

$$C(10,3) \cdot (\frac{1}{6})^3 \cdot (\frac{5}{6})^7 = 120 \cdot (\frac{1}{6})^3 \cdot (\frac{5}{6})^7 = \frac{390,625}{2,519,424} = 0.1550$$
.

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• Takes into account the fact that different outcomes may have different probabilities.

# Expected Value (cont'd)

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Expected value = 
$$\frac{1}{6} \times 1 + \frac{1}{6} \times 2 + \frac{1}{6} \times 3 + \frac{1}{6} \times 4 + \frac{1}{6} \times 5 + \frac{1}{6} \times 6$$
  
=  $\frac{1}{6} + \frac{2}{6} + \frac{3}{6} + \frac{4}{6} + \frac{5}{6} + \frac{6}{6}$   
=  $\frac{21}{6} = 3.5$ 

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• Why? Since the *n* events are equally likely, they each have probability 1/n.

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• Since two games per ticket, the probability of one ticket winning the jackpot is  $\frac{2}{45,057,474} = \frac{1}{22,528,737} = 4.4 \times 10^{-9}$ .

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- Since two games per ticket, the probability of one ticket winning the jackpot is  $\frac{2}{45,057,474} = \frac{1}{22,528,737} = 4.4 \times 10^{-9}$ .
- The expected amount you would win *if the ticket were free* would be

$$\mathsf{Prob}(\mathsf{win}) \times a + \mathsf{Prob}(\mathsf{lose}) \times 0 = \frac{a}{22,528,737}$$

- **Example:** How much can you expect to win (or lose) playing New York State Pick Six?
- Solution:
  - The total number of ways 6 balls can be chosen out of 59 is

$$C(59,6) = \frac{59!}{6!53!} = \frac{59 \times 58 \times 57 \times 56 \times 55 \times 54}{6 \times 5 \times 4 \times 3 \times 2}$$
  
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• What are the expected payoffs for one ticket for certain amounts of the grand jackpot?

Jackpot amount	Expected payoff
\$1,000,000	-\$0.955612
\$10,000,000	-\$0.556221
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\$100,000,000	\$3.43878

- **Example:** How much can you expect to win (or lose) playing New York State Pick Six?
- Solution (cont'd):
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• Break-even point: \$22,528,737