Counting
(Enumerative Combinatorics)

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Chance of winning ?

- What's the chances of winning New York Mega-million Jackpot
  - "just pick 5 numbers from 1 to 56, plus a mega ball number from 1 to 46, then you could win biggest potential Jackpot ever !"
    - If your 6-number combination matches winning 6-number combination (5 winning numbers plus the Mega Ball), then you win First prize jackpot.
  - There are many possible ways to choose 6-number
  - Only one of them is the winning combination…
  - If each 6-number combination is equally likely to be the winning combination …
    - Then the prob. of winning for any 6-number is 1/X

Counting

- How many bits are need to represent 26 different letters?

- How many different paths are there from a city to another, giving the road map?

Counting rule #1: just count it

- If you can count directly the number of outcomes, just count them.
- For example:
  - How many ways are there to select an English letter ?
    - 26 as there are 26 English letters
  - How many three digits integers are there ?
    - These are integers that have value ranging from 100 to 999.
    - How many integers are there from 100 to 999 ?
      - 999-100+1=900

Example of first rule

- How many integers lies within the range of 1 and 782 inclusive ?
  - 782, we just know this !
- How many integers lies within the range of 12 and 782 inclusive ?
  - Well, from 1 to 782, there are 782 integers
  - Among them, there are 11 number within range from 1 to 11.
  - So, we have 782-(12-1)=782-12+1 numbers between 12 and 782

Quick Exercise

- So the number of integers between two integers, S (smaller number) and L (larger number) is: L-S+1
- How many integers are there in the range 123 to 928 inclusive ?
- How many ways are there to choose a number within the range of 12 to 23, inclusive ?

A little more complex problems

- How many possible license plates are available for NY state ?
  - 3 letters followed by 4 digits (repetition allowed)
- How many 5 digits odd numbers if no digits can be repeated ?
- How many ways are there to seat 10 guests in a table?
- How many possible outcomes are there if draw 2 cards from a deck of cards ?
- Key: all above problems ask about # of combinations/ arrangements of people/digits/letters/…

How to count ?

- Count in a systematical way to avoid double-counting or miss counting
- Ex: to count num. of students present …
  - First count students on first row, second row, …
  - First count girls, then count boys

How to count (2)?

- Count in a systematical way to avoid double-counting or miss counting
- Ex: to buy a pair of jeans …
  - Styles available: standard fit, loose fit, boot fit and slim fit
  - Colors available: blue, black
  - How many ways can you select a pair of jeans ?

Use Table to organize counting

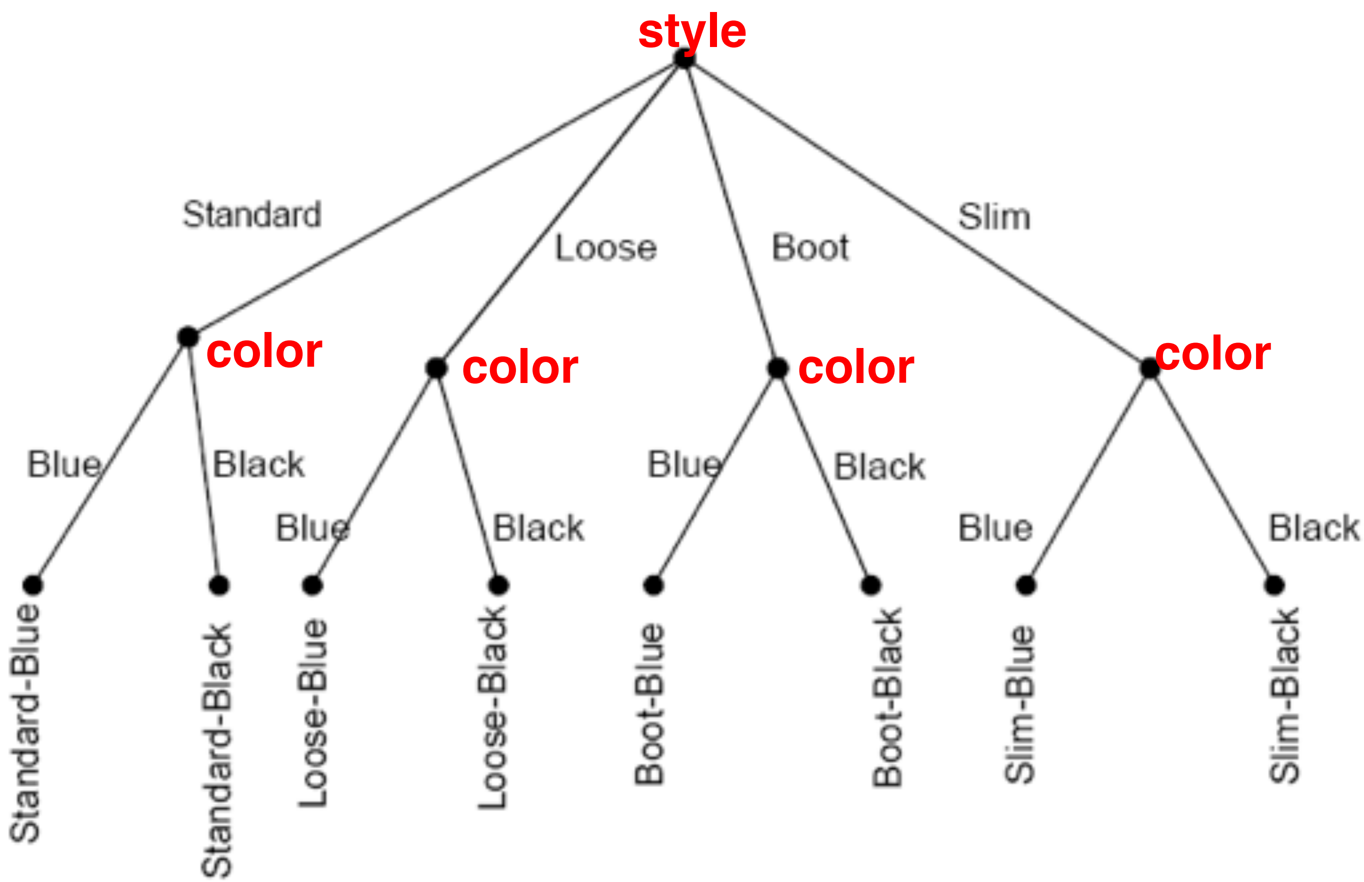
- Fix color first, and vary styles
  - Table is a nature solution

	Jean Style			
Color	**Standard**	**Loose**	**Boot**	**Slim**
Blue	Standard-Blue	Loose-Blue	Boot-Blue	Slim-Blue
Black	Standard-Black	Loose-Black	Boot-Black	Slim-Black

Table 1-1: Enumeration of Jean Configurations using a Table

- What if we can also choose size, Medium, Small or Large?
  - 3D table ?

Selection/Decision tree



Node: a feature/variable
Branch: a possible selection for the feature
Leaf: a configuration/combination

Let's try an example

- Enumerate all 3-letter words formed using letters from word "cat"
  - assuming each letter is used once.
- How would you do that ?
  - Choose a letter to put in 1st position, 2nd and 3rd position

Exercises

- Use a tree to find all possible ways to buy a car
  - Color can be any from {Red, Blue, Silver, Black}
  - Interior can be either leather or fiber
  - Engine can be either 4 cylinder or 6 cylinder
- How many different outcomes are there for a “best of 3” tennis match between player A and B?
  - Whoever wins 2 games win the match…

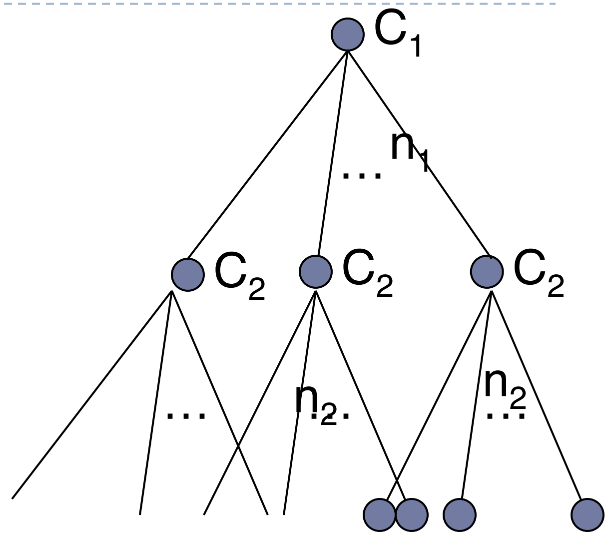
Terminology

- When buying a pair of jean, one can choose style and color
- We call style and color features/variables
- For each feature, there is a set of possible choices/options
  - For "style", the set of options is {standard, loose, boot, slim}
  - For "color", the set of options is {blue,black}
- Each configuration, i.e., standard-blue, is called an outcome/possibility

Outline on Counting

- Just count it
- Organize counting: table, trees
- Multiplication rule
- Permutation
- Combination
- Addition rule, Generalized addition rule
- Exercises

Counting rule #2: multiplication rule



- If we have two features/decisions C_1 and C_2
  - C_1 has n_1 possible outcomes/options
  - C_2 has n_2 possible outcomes/options
- Then total number of outcomes is $n_1 * n_2$
- In general, if we have k decisions to make:
  - C_1 has n_1 possible options
  - …
  - C_k has n_k possible options
  - then the total number of outcomes is $n_1 * n_2 * \ldots * n_k$.
- "AND rule":
  - You must make all the decisions…
  - i.e., C_1, C_2 , …, C_k must all occur

Jean Example

- Problem Statement
  - **Two decisions to make**: C_1=Chossing style, C_2=choosing color
  - Options for C_1 are {standard fit, loose fit, boot fit, slim fit}, $n_1=4$
  - Options for C_2 are {black, blue}, $n_2=2$
- To choose a jean, one must choose a style and choose a color
  - C_1 and C_2 must both occur, use multiplication rule
- So the total # of outcomes is $n_1*n_2=4*2=8$.

Coin flipping

- Flip a coin twice and record the outcome (head or tail) for each flip. How many possible outcomes are there ?
- Problem statement:
  - **Two steps for the experiment,** C_1= "first flip", C_2="second flip"
  - Possible outcomes for C_1 is {H, T}, $n_1=2$
  - Possible outcomes for C_2 is {H,T}, $n_2=2$
  - C_1 occurs **and** C_2 occurs: total # of outcomes is $n_1*n_2=4$

License Plates

- Suppose license plates starts with two different letters, followed by 4 letters or numbers (which can be the same). How many possible license plates ?
- Steps to choose a license plage:
  - Pick two different letters AND pick 4 letters/numbers.
  - C_1: Pick a letter
  - C_2: Pick a letter different from the first
  - C3,C4,C5,C6: Repeat for 4 times: pick a number or letter
- Total # of possibilities:
  - 26*25*36*36*36*36 = 1091750400
- Note: the num. of options for a feature/variable might be affected by previous features

Exercises:

- In a car racing game, you can choose from 4 difficulty level, 3 different terrains, and 5 different cars, how many different ways can you choose to play the game ?
- How many ways can you arrange 10 different numbers (i.e., put them in a sequence)?

Relation to other topics

- It might feel like that we are topics-hopping
  - Set, logic, function, relation …
- Counting:
  - What is being counted ?
    - A finite set, i.e., we are evaluate some set's cardinality when we tackle a counting problem
  - How to count ?
    - So rules about set cardinality apply !
    - Inclusion/exclusion principle
    - Power set cardinality
    - Cartisian set cardinality

Learn new things by reviewing old…

- Sets cardinality: number of elements in set
  - |AxB| = |A| x |B|
  - The number of diff. ways to pair elements in A with elements in B, i.e., |AxB|, equals to |A| x |B|
- Example
  - A={standard, loose, boot}, the set of styles
  - B={blue, black}, the set of colors
  - AxB= {(standard, blue), (standard, black), (loose, blue), (loose, black), (boot, blue), (boot, black)}, the set of different jeans
  - |AxB|: # of different jeans we can form by choosing from A the style, and from B the color

Let’s look at more examples…

Seating problem

- How many different ways are there to seat 5 children in a row of 5 seats?
  - Pick a child to sit on first chair
  - Pick a child to sit on second chair
  - Pick a child to sit on third chair
  - …
    - The outcome can be represented as an ordered list: e.g. Alice, Peter, Bob, Cathy, Kim
  - By multiplication rule: there are 5*4*3*2*1=120 different ways to sit them.
  - Note, "Pick a chair for 1st child" etc. also works

Job assignment problem

- How many ways to assign 5 diff. jobs to 10 volunteers, assuming each person takes at most one job, and one job assigned to one person ?
  - Pick one person to assign to first job: 10 options
  - Pick one person to assign to second job: 9 options
  - Pick one person to assign to third job: 8 options
  - …
  - In total, there are 10*9*8*7*6 different ways to go about the job assignments.

Permutation

- Some counting problems are similar
  - How many ways are there to arrange 6 kids in a line ?
  - How many ways to assign 5 jobs to 10 volunteers, assuming each person takes at most one job, and one job assigned to one person ?
  - How many different poker hands are possible, i.e. drawing five cards from a deck of card where order matters ?

Permutation

- A permutation of objects is an arrangement where order/position matters.
  - Note: “arrangement” implies each object cannot be picked more than once.
  - Seating of children
    - Positions matters: Alice, Peter, Bob, Cathy, Kim is different from Peter, Bob, Cathy, Kim, Alice
  - Job assignment: choose 5 people out of 10 and arrange them (to 5 jobs)
  - Select a president, VP and secretary from a club

Permutations

- Generally, consider choosing r objects out of a total of n objects, and arrange them in r positions.

n objects
(n gifts)

…

_____ _____ _____ … _____ _____

1 2 3 r-1 r

r positions (r behaving Children)

Counting Permutations

- Let P(n,r) be the number of permutations of r items chosen from a total of n items, where r≤n
  - n objects and r positions
    - Pick an object to put in 1st position, # of ways: n
    - Pick an object to put in 2nd position, # of ways: n-1
    - Pick an object to put in 3rd position, # of ways: n-2
    - …
    - Pick an object to put in r-th position, # of ways: n-(r-1)
    - By multiplication rule,

$$P(n,r) = n \cdot (n-1) \cdot (n-2) \ldots (n-r+1)$$

Note: factorial

- n! stands for "n factorial", where n is positive integers, is defined as

$$n! = n \cdot (n-1) \cdot \ldots 3 \cdot 2 \cdot 1$$

- Now

$$P(n,r) = n \cdot (n-1) \ldots (n-r+1)$$

$$= \frac{n \cdot (n-1) \ldots (n-r+1) \cdot (n-r) \ldots 2 \cdot 1}{(n-r) \ldots 2 \cdot 1}$$

$$= \frac{n!}{(n-r)!}$$

Examples

- How many five letter words can we form using distinct letters from set {a,b,c,d,e,f,g,h} ?
  - It’s a permutation problem, as the order matters and each object (letter) can be used at most once.
  - P(8,5)

Examples

- How many ways can one select a president, vice president and a secretary from a class of 28 people, assuming each student takes at most one position ?
  - A permutation of 3 people selecting from 28 people: P(28,3)=28*27*26

Exercises

- What does P(10,2) stand for ? Calculate P(10,2).
- How about P(12,12)?
- How many 5 digits numbers are there where no digits are repeated and 0 is not used ?

Examples: die rolling

- If we roll a six-sided die three times and record results as an ordered list of length 3
  - How many possible outcomes are there ?
    - 6*6*6=216
  - How many possible outcomes have different results for each roll ?
    - 6*5*4
  - How many possible outcomes do not contain 1 ?
    - 5*5*5=125

Combinations

- Many selection problems do not care about position/order
  - from a committee of 3 from a club of 24 people
  - Santa select 8 million toys from store
  - Buy three different fruits
- Combination problem: select r objects from a set of n distinct objects, where order does not matter.

Combination formula

- C(n,r): number of combinations of r objects chosen from n distinct objects (n>=r)
  - Ex: ways to buy 3 different fruits, choosing from apple, orange, banana, grape, kiwi: C(5,3)
  - Ex: ways to form a committee of two people from a group of 24 people: C(24,2)
  - Ex: Number of subsets of {1,2,3,4} that has two elements: C(4,2)
- Next: derive formula for C(n,r)

Deriving Combination formula

- How many ways are there to form a committee of 2 for a group of 24 people ?
  - Order of selection doesn't matter
- Let's try to count:
  - There are 24 ways to select a first member
  - And 23 ways to select the second member
  - So there are 24*23=P(24,2) ways to select two peoples in sequence
- In above counting, each two people combination is counted twice
  - e.g., For combination of Alice and Bob, we counted twice: (Alice, Bob) and (Bob, Alice).
- To delete overcounting
  - P(24,2)/2

General formula

- when selecting r items out of n distinct items
  - If order of selection matters, there are P(n,r) ways
  - For each combination (set) of r items, they have been counted many times, as they can be selected in different orders:
    - For r items, there are P(r,r) different possible selection order
      - e.g., {Alice, Bob} can be counted twice: (Alice, Bob) and (Bob, Alice). (if r=2)
    - Therefore, each set of r items are counted P(r,r) times.
  - The # of combinations is:

$$C(n,r) = \frac{P(n,r)}{P(r,r)} = \frac{n!/(n-r)!}{r!/(r-r)!} = \frac{n!}{r!(n-r)!}$$

A few exercise with C(n,r)

$$C(n,r) = \frac{n!}{r!(n-r)!}$$

- Calculate C(7,3)

$$C(7,3) = \frac{7!}{(7-3)!3!} = \frac{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{4\cdot 3\cdot 2\cdot 1\cdot 3\cdot 2\cdot 1} = \frac{7\cdot 6\cdot 5}{3\cdot 2\cdot 1} = 35$$

- What is C(n,n) ? How about C(n,0)?
- Show C(n,r)=C(n,n-r).

Committee Forming

- How many different committees of size 7 can be formed out of 20-person office ?
  - C(20,7)
  - Three members (Mary, Sue and Tom) are carpooling. How many committees meet following requirement ?
    - All three of them are on committee:C(20-7,4)
    - None of them are on the committee:C(20-7,7)

Outline on Counting

- Just count it
- Organize counting: table, trees
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Set Related Example

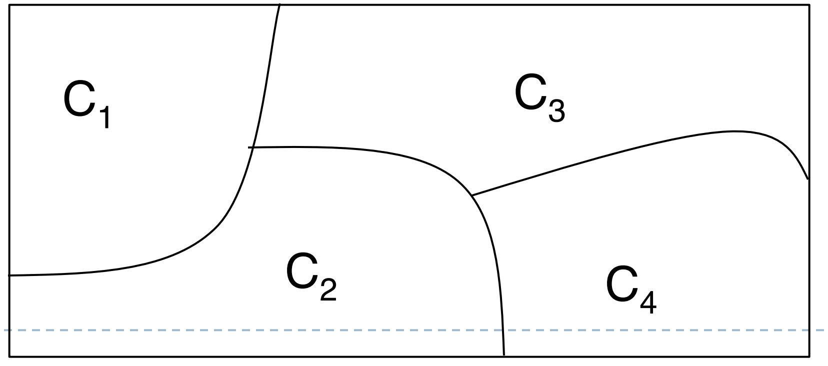
- How many subsets of {1,2,3,4,5,6} have 3 elements ?
  - C(6,3)
- How many subsets of {1,2,3,4,5,6} have an odd number of elements ?
  - Either the subset has 1, or 3, or 5 elements.
  - C(6,1)+C(6,3)+C(6,5)

Knapsack Problem

- There are n objects
  - The i-th object has weight w_i, and value v_i
- You want to choose objects to take away, how many possible ways are possible ?
  - $2*2*\ldots*2=2^n$
  - $C(n,0)+C(n,1)+\ldots+C(n,n)$
- Knapsack problem:
  - You can only carry W pound stuff
  - What shall you choose to maximize the value ?

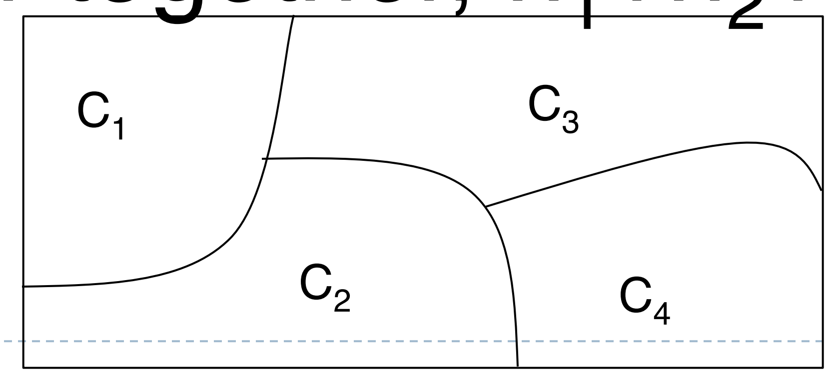
Addition Rule

- If the events/outcomes that we count can be decomposed into k cases $C_1, C_2, \ldots, C_k$, each having $n_1, n_2, \ldots n_k$, possible outcomes respectively,
  - (either C_1 occurs, or C_2 occurs, or C_3 occurs, …. or C_k occurs)
- Then the total number of outcomes is $n_1+n_2+\ldots+n_k$.



Key to Addition Rule

- Decompose what you are counting into simpler, easier to count scenarios, C_1, C_2, …, C_k
- Count each scenario separately, $n_1,n_2,\ldots,n_k$
- Add the number together, $n_1+n_2+\ldots+n_k$



Examples: die rolling

- If we roll a six-sided die three times and record results as an ordered list of length 3
  - How many of the possible outcomes contain exactly one 1, e.g. 1,3,2 or, 3,2,1, or 5,1,3 ?
    - Let's try multiplication rule by analyzing what kind of outcomes satisfy this ?
    - First roll: 6 possible outcomes
    - Second roll: # of outcomes ?
      - If first roll is 1, second roll can be any number but 1
      - If first roll is not 1, second roll can be any number
    - Third roll: # of outcomes ??

Examples: die rolling

- If we roll a six-sided die three times and record results as an ordered list of length 3
  - how many of the possible outcomes contain exactly one 1 ?
  - Let's try to consider three different possibilities:
    - The only 1 appears in first roll, C_1
    - The only1 appears in second roll, C_2
    - The only1 appears in third roll, C_3
  - We get exactly one 1 if C_1 occurs, or C_2 occurs, or C_3 occurs
  - Result: 5*5+5*5+5*5=75

Examples: die rolling

- If we roll a six-sided die three times, how many of the possible outcomes contain exactly one 1 ? Let's try another approach :
  - First we select where 1 appears in the list
    - 3 possible ways
  - Then we select outcome for the first of remaining positions
    - 5 possible ways
  - Then we select outcome for the second of remaining positions
    - 5 possible ways

Result: 3*5*5=75

Example: Number counting

- How many positive integers less than 1,000 consists only of distinct digits from {1,3,7,9} ?
- To make such integers, we either
  - Pick a digit from set {1,3,7,9} and get an one-digit integer
  - Take 2 digits from set {1,3,7,9} and arrange them to form a two-digit integer
    - permutation of length 2 with digits from {1,3,7,9}.
  - Take 3 digits from set {1,3,7,9} and arrange them to form a 3-digit integer
    - a permutation of length 3 with digits from {1,3,7,9}.

Example: Number Counting

- Use permutation formula for each scenario (event)
  - # of one digit number: P(4,1)=3
  - # of 2 digit number: P(4,2)=4*3=12
  - # of 3 digit number: P(4,3)=4*3*2=24
- Use addition rule, i.e., “OR” rule
  - Total # of integers less than 1000 that consists of {1,3,7,9}: 3+12+24=39

Example: computer shipment

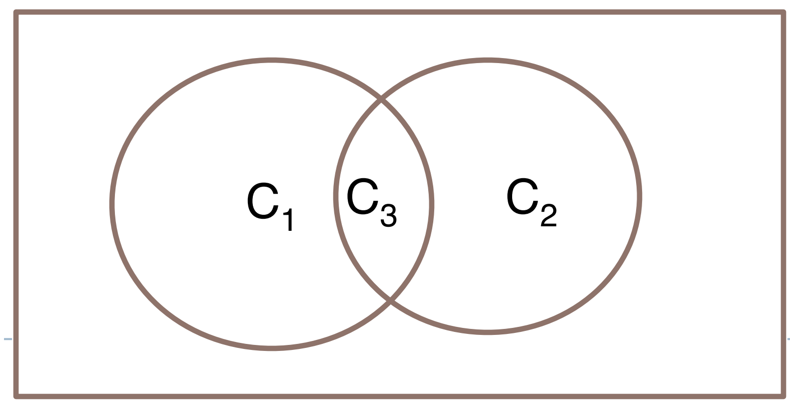
- Suppose a shipment of 100 computers contains 4 defective ones, and we choose a sample of 6 computers to test.
  - How many different samples are possible ?
    - C(100,6)
  - How many ways are there to choose 6 computers if all four defective computers are chosen?
    - C(4,4)*C(96,2)
  - How many ways are there to choose 6 computers if one or more defective computers are chosen?
    - C(4,4)*C(96,2)+C(4,3)*C(96,3)+C(4,2)*C(96,4)+C(4,1)*C(96,5)
    - C(100,6)-C(96,6)

Generalized addition rule

- If we roll a six-sided die three times how many outcomes have exactly one 1 or exactly one 6 ?
  - How many have exactly one 1 ?
    - 3*5*5
  - How many have exactly one 6 ?
    - 3*5*5
  - Just add them together ?
    - Those have exactly one 1 and one 6 have been counted twice
    - How many outcomes have exactly one 1 and one 6 ?
      - C(4,1)P(3,3)=4*3*2

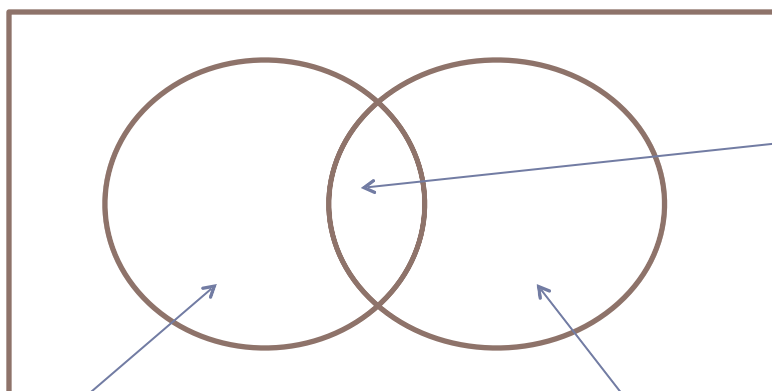
Generalized addition rule

- If we have two choices C_1 and C_2,
  - C_1 has n_1 possible outcomes,
  - C_2 has n_2 possible outcomes,
  - C_1 and C_2 both occurs has n_3 possible outcomes
- then total number of outcomes for C_1 or C_2 occurring is $n_1+n_2-n_3$.



Generalized addition rule

- If we roll a six-sided die three times how many outcomes have exactly one 1 or exactly one 6 ?
  - 3*5*5+3*5*5-3*2*4



Outcomes that have exactly one 1 and one 6, such as (1,2,6), (3,1,6)

Outcomes that have exactly one 1, such as (1,2,3), (1,3,6), (2,3,1)

Outcomes that have exactly one 6, such as (2,3,6), (1,3,6), (1,1,6)

Example

- A class of 15 people are choosing 3 representatives, how many possible ways to choose the representatives such that Alice or Bob is one of the three being chosen? Note that they can be both chosen.

Summary: Counting

- How to tackle a counting problem?

1. Some problems are easy enough to just count it, by enumerating all possibilities.
2. Otherwise, does multiplication rule apply, i.e., a sequence of decisions is involved, each with a certain number of options?

Summary: Counting

- How to tackle a counting problem?

  3. Otherwise, is it a permutation problem ?

Summary: Counting (cont'd)

- How to tackle a counting problem?

4. Is it a combination problem ?

Summary: Counting (cont'd)

- How to tackle a counting problem?
  5. Can we break up all possibilities into different situations/cases, and count each of them more easily?

Summary: Counting (cont'd)

- How to tackle a counting problem?
  - Often you use multiple rules when solving a particular problem.
    - First step is hardest.
  - Practice makes perfect.

Exercise

- A class has 15 women and 10 men. How many ways are there to:
  - choose one class member to take attendance?
  - choose 2 people to clean the board?
  - choose one person to take attendance and one to clean the board?
  - choose one to take attendance and one to clean the board if both jobs cannot be filled with people of same gender?
  - choose one to take attendance and one to clean the board if both jobs must be filled with people of same gender?

Exercise

- A Fordham Univ. club has 25 members of which 5 are freshman, 5 are sophomores, 10 are juniors and 5 are seniors. How many ways are there to
  - Select a president if freshman is illegible to be president?
  - Select two seniors to serve on College Council?
  - Select 8 members to form a team so that each class is represented by 2 team members?

Cards problems

- A deck of cards contains 52 cards.
  - four suits: clubs, diamonds, hearts and spades
  - thirteen denominations: 2, 3, 4, 5, 6, 7, 8, 9, 10, J(ack), Q(ueen), K(ing), A(ce).
  - begin with a complete deck, cards dealt are not put back into the deck
  - abbreviate a card using denomination and then suit, such that 2H represents a 2 of Hearts.

How many different flush hands?

- A poker player is dealt a hand of 5 cards from a freshly mixed deck (order doesn't matter).
  - How many ways can you draw a flush? Note: a flush means that all five cards are of the same suit.

More Exercises

- A poker player is dealt a hand of 5 cards from a freshly mixed deck (order doesn't matter).
  - How many different hands have 4 aces in them?
  - How many different hands have 4 of a kind, i.e., you have four cards that are the same denomination?
  - How many different hands have a royal flush (i.e., contains an Ace, King, Queen, Jack and 10, all of the same suit)?

Shirt-buying Example*

- A shopper is buying three shirts from a store that stocks 9 different types of shirts. How many ways are there to do this, assuming the shopper is willing to buy more than one of the same shirt?
  - There are only the following possibilities,
    - She buys three of the same type:
    - Or, she buys three different type of shirts:
    - Or, she buy two of the same type shirts, and one shift of another type:
  - Total number of ways: 9+C(9,3)+9*8

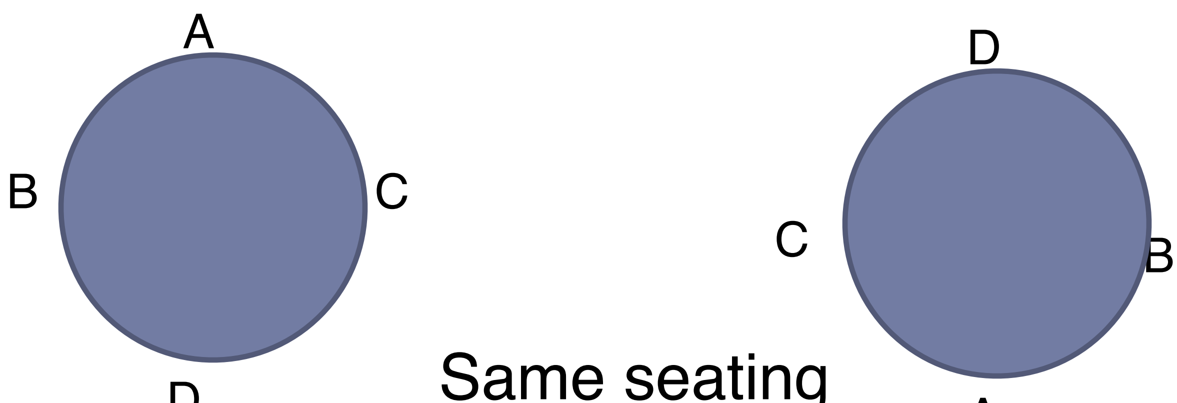
9

C(9,3)

9*8

Round table seating

- How many ways are there to arrange four children (A,B,C,D) to sit along a round table, suppose only relative position matters ?



- As only relative position matters, let's first fix a child, A, how many ways are there to seat B,C,D relatively to A?
  - P(3,3)

Some challenges

- In how many ways can four boys and four girls sit around a round table if they must alternate boy-girl-boy-girl?
  - Hints:
    1. fix a boy to stand at a position
    2. Arrange 3 other boys
    3. Arrange 4 girls

Some challenges

- A bag has 32 balls – 8 each of orange, white, red and yellow. All balls of the same color are indistinguishable. A juggler randomly picks three balls from the bag to juggle. How many possible groupings of balls are there?
  - Hint: cannot use combination formula, as balls are not all distinct as balls of same color are indistinguishable