Probability

Dr. Zhang
Fordham Univ.

Probability: outline

- Introduction
  - Experiment, event, sample space
  - Probability of events
- Calculate Probability
  - Through counting
  - Sum rule and general sum rule
  - Product rule and general product rule
  - Conditional probability
- Probability distribution function
- Bernoulli process

Start with our intuition

- What's the probability/odd/chance of
  - getting "head" when tossing a coin?
    - 0.5 if it's a fair coin.
  - getting a number larger than 4 with a roll of a die ?
    - 2/6=1/3, if the die is fair one
  - drawing either the ace of clubs or the queen of diamonds from a deck of cards (52) ?
    - 2/52

Our approach

- Divide # of outcomes of interests by total # of possible outcomes
- Hidden assumptions: different outcomes are equally likely to happen
  - Fair coin (head and tail)
  - Fair dice
  - Each card is equally likely to be drawn

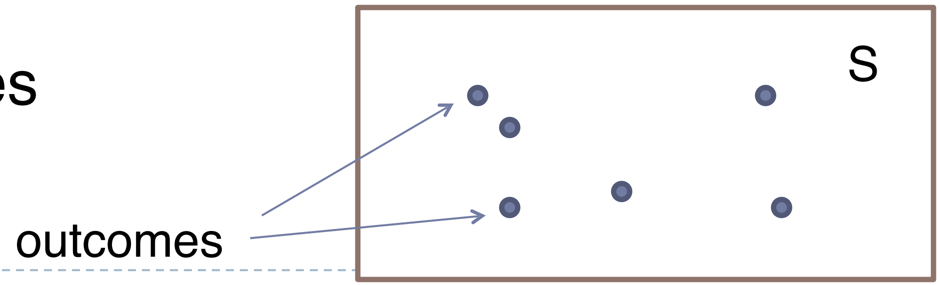
Another example

- In your history class, there are 24 people. Professor randomly picks 2 students to quiz them. What's the probability that you will be picked ?
  - Total # of outcomes?
  - # of outcomes with you being picked?

Terminology: Experiment, Sample Space

- **Experiment:** action that have a measurable outcome, e.g., :
  - Toss coins, draw cards, roll dices, pick a student from the class
- **Outcome**: result of the experiment
  - For tossing a coin, outcomes are getting a head, H, or getting a tail, T.
  - For tossing a coin twice, outcomes are HH, HT, TH, or TT.
  - When picking two students to quiz, outcomes are subsets of size two
- **Sample space of an experiment**: the set that contains all possible outcomes of the experiment, denoted by S.
  - Tossing a coin once: sample space is {H,T}
  - Rolling a dice: sample space is {1,2,3,4,5,6} .
  - …
- S is universe set as it includes all possible outcomes

Venn Diagram

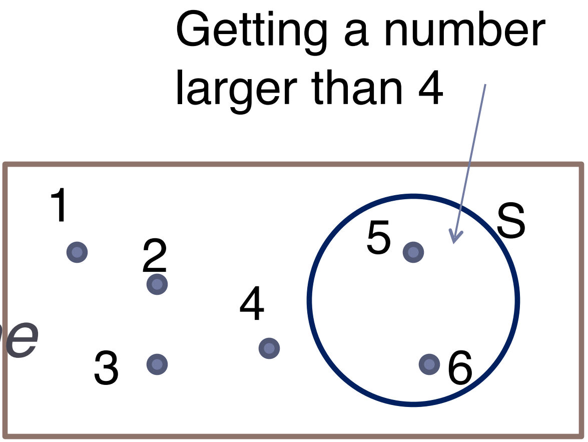


Example

- When the professor picks 2 students (to quiz) from a class of 24 students…
  - What's the sample space?
    - All the different outcomes of picking 2 students out of 24
  - How many possible outcomes are there?
    - That is same as asking "How many different outcomes are possible when picking 2 students from a class of 24 students?"
    - It's a counting problem!
    - C(24,2): order does not matter

Events

- **Event** : a subset of sample space S
  - "getting number larger than 4" is an event for rolling a die experiment
  - "you are picked to take quiz" is an event for picking two students to quiz
  - An event is said to occur if *an outcome in the subset occurs*
- Some special events:
  - Elementary event: event that contains exactly one outcome
  - { }: null event
  - S: sure event



Rolling a die experiment

"Getting a number larger than 4" occurs if 5 or 6 occurs

(Discrete) Probability

- If sample space S is a finite set of equally likely outcomes, then the probability of event E occurs, Pr(E) is defined as:

$$\Pr(E) = \frac{|E|}{|S|}$$

  - Likelihood or chance that the event occurs, e.g., if one repeats experiment for many times, frequency that the event happens
  - Note: sometimes we write P(E). It should be clear from context whether P stands for “probability” or “power set”
  - This captures our intuition of probability.

Example

- When the professor picks 2 students (to quiz) from a class of 24 students…
  - What's the sample space?
    - All the different outcomes of picking 2 students out of 24
  - How many possible outcomes are there?
    - |S| = C(24,2)
  - Event of interest: you are one of the two being picked
    - How many outcomes in the event ? i.e., how many outcomes have you as one of the two picked ?
    - |E| = C(1,1) C(23,1)
  - Prob. of you being picked:

$$\Pr(E) = \frac{|E|}{|S|} = \frac{23}{C(24,2)} = \frac{1}{12}$$

Probability: outline

- Introduction
  - Experiment, event, sample space
  - Probability of events
- Calculate Probability
  - through counting
  - Sum rule and general sum rule
  - Product rule and general product rule

Calculate probability by counting

- If sample space S is a finite set of equally likely outcomes, then the probability of event E occurs is:

$$\Pr(E) = \frac{|E|}{|S|}$$

- To calculate probability of an event for an experiment,
  - Identify sample space of the experiment, S, i.e., what are the possible outcomes ?
  - Count number of all possible outcomes, i.e., cardinality of sample space, ISI
  - Count number of outcomes in the event, i.e., cardinality of event, I EI
  - Obtain prob. of event as Pr(E)=IEI/ISI

Example: Toss a coin

- if we toss a coin once, we either get a tail or get a head.
  - sample space can be represented as {Head, Tail} or simply {H,T}.
  - The event of getting a head is the set {H}.
    - Prob ({H})=|{H}| / | {H,T}| = 1/2
  - The event of getting a tail is the set {T}
  - The event of getting a head or tail is the set {H,T}, i.e., the whole sample space

Example: coin tossing

- If we toss a coin 3 times, what's the probability of getting three heads?
  - Sample space, S: {HHH, HHT, ..., TTT}
  - There are 2x2x2=8 possible outcomes, |S|=8
  - There is one outcome that has three heads, HHH. |E|=1
  - So probability of getting three head is: |E|/|S|=1/8
- What's the probability of getting same results on last two tosses, E ?
  - Outcomes in E are HHH, THH, HTT, TTT, so |E|=4
  - Or how many outcomes have same results on last two tosses?
    - 2*2=4
  - Prob. of getting same results on last two tosses: 4/8=1/2.

Example: poke cards

- When we draw a card from a standa cards (52 cards, 13 cards for each suits).
  - Sample space is:
    - All 52 cards
  - Num. of outcomes that getting an ace is:
    - lEl=4
  - Probability of getting an ace is:
    - lEl/lSl=4/52
  - Probability of getting a red card or an ace is:
    - lEl=26 red cards+2 black ace cards=28
    - Pr (E)=28/52

Example: dice rolling

- If we roll a pair of dice and record sum of face-up numbers, what's the probability of getting a 10 ?
  - The sum of face-up numbers can be any of the following: 2,3,4,5,6,7,8,9,10,11,12.
    - S={2,3,4,5,6,7,8,9,10,11,12}
  - So the prob. of getting a 10 is 1/11
    - Pr(|E|)=|E|/|S|=1/11
  - Any problem in above calculation?
    - Are all outcomes in sample space equally like to happen ?
    - No, there are two ways to get 10 (by getting 4 and 6, or getting 5 and 5), there are just one way to get 2 (by getting 1 and 1),…

Example: dice rolling (cont'd)

- If we roll a pair of dice and record sum of face-up numbers, what's the probability of getting a 10 ?
  - Represent outcomes as ordered pair of numbers, i.e. (1,5) means getting a 1 and then a 5
  - How many outcomes are there ? i.e., ISI=?
    - 6*6
  - Event of getting a 10 is: {(4,6),(5,5),(6,4)}
  - Prob. of getting 10 is: 3/(6*6)

Example: counting outcomes

- Drawing two cards from the top of a deck of 52 cards, the probability that two cards having same suit ?
  - Sample space S:
    - |S|=52*51 , 52 choices for first draw, 51 for second
  - Event that two cards have same value, E:
    - |E|=52*12, 52 choices for first draw, 12 for second (from remaining 12 cards of same suit as first card)
  - Pr (E)=|E|/|S|=(52*12)/(52*51)=12/51

Example: card game

- At a party, each card in a standard deck is torn in half and both haves are placed in a box. Two guests each draw a half-card from the box. What's the probability that they draw two halves of the same card ?
  - Size of sample space, i.e., how many ways are there to draw two from the 52*2 half-cards ?
    - 104*103
  - How many ways to draw two halves of same card?
    - 104*1
  - Prob. that they draw two halves of same card
    - 104/(104*103)=1/103.

NY Jackpot Lottery

- "pick 5 numbers from 1 to 56, plus a mega ball number from 1 to 46,"
  - If your 5-number combination matches winning 5-number combination, and mega ball number matches the winning Mega Ball, then you win !
  - Order for the 5 numbers does not matter.
- Sample space: all different ways one can choose 5-number combination, and a mega ball number
  - |S|= ?
- Winning event contains the single outcome in sample space, i.e., the winning comb. and mega ball number
  - |E|=1, Pr(E)=1/|S|=

Probability of Winning
Lottery Game

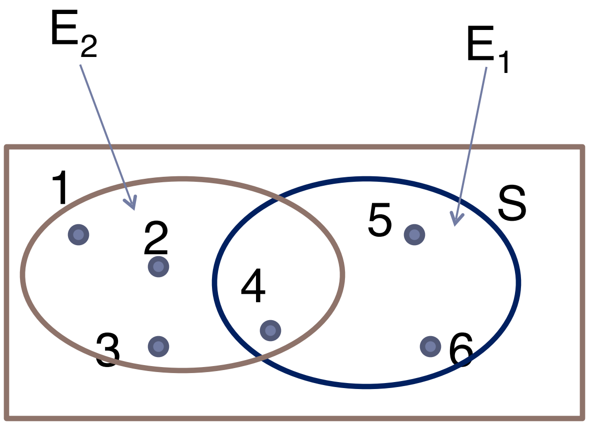
- In one lottery game, you pick 7 distinct numbers from {1,2,…,80}.
- On Wednesday nights, someone's grandmother draws 11 numbered balls from a set of balls numbered from {1,2,…80}.
- If the 7 numbers you picked appear among the 11 drawn numbers, you win.
- What is your probability of winning?
- Questions:
  - What is the experiment, sample space ?
  - What is the winning event ?

Probability: outline

- Introduction
  - Experiment, event, sample space
  - Probability of events
- Calculate Probability through counting
  - Examples, exercises
- Sum rule and general sum rule
  - Examples and exercises
- Product rule and general product rule
  - Conditional probability

Events are sets

- **Event of an experiment**: any subset of sample space S, e.g.
- Events are sets, therefore all set operations apply to events
  - Union: $E_1 \cup E_2$
    - E_1 or E_2 occurs
  - Intersection: $E_1 \cap E_2$
    - E_1 and E_2 both occurs
  - Complements:
    $E^c = U - E = S - E$
    - E does not occur



Die rolling experiment

E_1: getting a number greater than 3
E_2: getting a number smaller than 5

Properties of probability

- Recall: For an experiment, if its sample space S is a finite set of equally likely outcomes, then the probability of event E occurs, Pr(E) is given by :

$$\Pr(E) = \frac{|E|}{|S|}$$

- For any event E, we have 0≤ |E|≤|S|, so
  - $0 \le \Pr(E) \le 1$
  - Extreme cases: P(S)=1, P({})=0
- Sometimes, counting |E| (# of outcomes in event E) is hard
  - And it's easier to count number of outcomes that are not in E, i.e., $|E^c|$

$$\Pr(E) = \frac{|E|}{|S|} = \frac{|S| - |E^c|}{|S|} = \frac{|S|}{|S|} - \frac{|E^c|}{|S|} = 1 - \Pr(E^c)$$

Tossing a coin 3 times

- What's the probability of getting at least one head ?
  - How large is our sample space ?
    - 2*2*2=8
  - How many outcomes have at least one head ???
    - How many outcomes has no head ?
    - # of outcomes that have at least one head is:

      2*2*2-1=7
  - Prob. of getting at least one head is 7/8
  - Alternatively,

$$\Pr(E) = 1 - \Pr(E^c) = 1 - 1/8$$

Example: Birthday problem

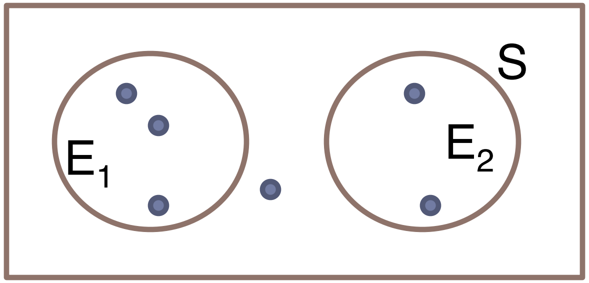
- What is the probability that in one class of 8 students, there are at least two students having birthdays in the same month (E), assuming each student is equally likely to have a birthday in the 12 months ?
  - Sample space: 12^8
  - Consider E^c :all students were born in different months
    - Outcomes that all students were born in diff. months is a permutation of 12 months to 8 students, therefore total # of outcomes in E^c: P(12,8)
  - $Pr(E^c) = P(12,8)/12^8$
  - Answer: $Pr(E)=1-Pr(E^c)=1 - P(12,8)/12^8$

Exercise:

- A class with 14 women and 16 men are choosing 6 people randomly to take part in an event
- What's the probability that at least one woman is selected?
- What's the probability that at least 3 women are selected?

Disjoint event

- Two events E_1, E_2 for an experiment are said to be disjoint (or mutually exclusive) if they cannot occur simultaneously, i.e. $E_1 \cap E_2 = \phi$

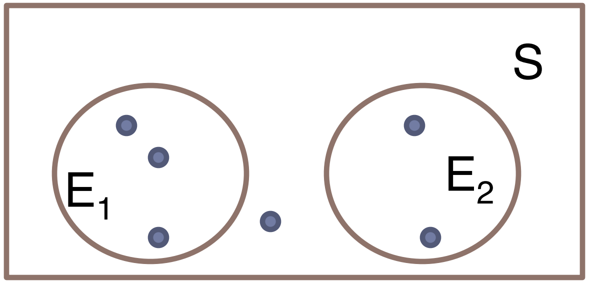


- tossing a die once
  - "getting a 3" and "getting a 4"
    - disjoint
  - "getting a 3" and "not getting a 6"
    - not disjoint
- tosses of a die twice
  - "getting a 3 on the first roll" and "getting a 4 on the second roll"
    - not disjoint.

Addition rule of probability

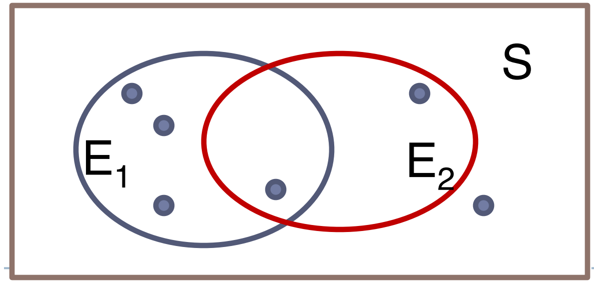
- if E_1 are E_2 are disjoint,

$$P(E_1 \cup E_2) = P(E_1) + P(E_2)$$



- Generally,

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$



Applying addition Rule

- When you toss a coin 5 times, what's the probability of getting an even number of heads?
  - Getting an even number of heads = "getting 0 heads" or "getting 2 heads" or "getting 4 heads"
  - i.e., $E = E_0 \cup E_2 \cup E_4$
  - It's like addition rule for counting. We decompose the event into smaller events which are easier to count, and each smaller events have no overlap.
  - So $\Pr(E)=\Pr(E_0)+\Pr(E_2)+\Pr(E_4)$
  - Try to find $\Pr(E_0)$, $\Pr(E_2)$, and $\Pr(E_4)$…

Example of applying rules

- The professor is randomly picking 3 students from a class of 24 students to quiz. What's the prob. that you or your best friend (or both) is selected?
  - Calculate it directly:
    - |E|: how many ways are there to pick 3 students so that either you or your best friend or both of you are selected.
  - Or: Let E1 be the event that you are selected, E2: your best friend is selected

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

    - Is $E_1 \cap E_2$ an empty event?

Exercise: addition rule

- You draw 2 cards randomly from a deck of 52 cards, what's the probability that the 2 cards have the same value or are of the same color ?

- You draw 2 cards randomly from a deck of 52 cards, what's the probability that the 2 cards have the same value or are of the same suit ?

Probability: outline

- Introduction
  - Experiment, event, sample space
  - Probability of events
- Calculate Probability through counting
  - Examples, exercises
- Sum rule and general sum rule
  - Examples and exercises
- Product rule and general product rule
  - Conditional probability

Independent event

- Two events, E_1 and E_2, are said to be independent if occurrence of E_1 event is not influenced by occurrence (or non-occurrence) of E_2, and vice versa
- Tossing of a coin for 10 times
  - "getting a head on first toss", and "getting a head on second toss"
  - "getting 9 heads on first 9 tosses", "getting a tail on 10^{th} toss"

Independent event

- A drawer contains 3 red paperclips, 4 green paperclips, and 5 blue paperclips. One paperclip is taken from the drawer and then replaced. Another paperclip is taken from the drawer.
  - E1: the first paperclip is red
  - E2: the second paperclip is blue
  - E1 and E2 are independent
- Typically, independent events refer to
  - Different and independent aspects of experiment outcome

- A drawer contains 3 red paperclips, 4 green paperclips, and 5 blue paperclips. One paperclip is taken from the drawer and not put back in the drawer. Another paperclip is taken from the drawer.
  - E1: the first paperclip is red
  - E2: the second paperclip is blue
  - Are E1 and E2 independent?
    - If E1 happens,

Independent event: example

- Choosing a committee of three people from a club with 8 men and 12 women, "the committee has a woman" (E_1) and "the committee has a man" (E_2)
  - If E_1 occurs, …
  - If E_1 does not occur (i.e., the committee has no woman), then E_2 occurs for sure
  - So, E_1 and E_2 are not independent

Product rule (Multiplication rule)

- If E_1 and E_2 are independent events in a given experiment, then the probability that both E_1 and E_2 occur is the product of $P(E_1)$ and $P(E_2)$:

$$P(E_1 \cap E_2) = P(E_1) \cdot P(E_2)$$

  - Prob. of getting two heads in two coin flips
    - E_1: getting head in first flip, $P(E_1)=1/2$
    - E_2: getting head in second flip, $P(E_2)=1/2$
    - E_1 and E_2 are independent

$$P(E_1 \cap E_2) = P(E_1) \cdot P(E_2) = 1/4$$

Independent event

- Pick 2 marbles one by one randomly from a bag of 10 black marbles and 10 blue marbles, with replacement (i.e., first marble drawn is put back to bag)
  - Prob. of getting a black marble first time and getting a blue marble second time ?
  - E_1: getting a black marble first time
  - E_2: getting a blue marble second time
  - E_1 and E_2 are independent (because of replacement)

$$P(E_1 \cap E_2) = P(E_1) \cdot P(E_2) = \frac{10}{20} * \frac{10}{20} = 0.25$$

What if no replacement ?

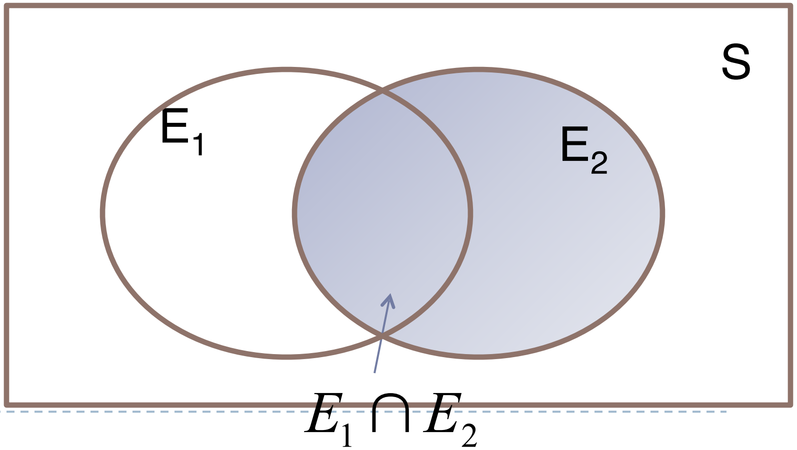
- Pick 2 marbles one by one randomly from a bag of 10 black marbles and 10 blue marbles, without replacement (i.e., first marble drawn is not put back)
  - Prob. of getting a black marble first, and getting a blue marble second time ?
  - E_1: getting a black marble in first draw
  - E_2: getting a blue marble in second draw
  - Are E_1 and E_2 independent ?
    - If E_1 occurs, prob. of E_2 occurs is 10/19
    - If E_1 does not occurs, prob. of E_2 occurs is: 9/19
  - So, they are not independent

Conditional Probability

- Probability of E_1 given that E_2 occurs, P ($E_1|E_2$), is given by:

$$\Pr(E_1 | E_2) = \frac{|E_1 \cap E_2|}{|E_2|} = \frac{|E_1 \cap E_2| / |S|}{|E_2| / |S|} = \frac{\Pr(E_1 \cap E_2)}{\Pr(E_2)}$$

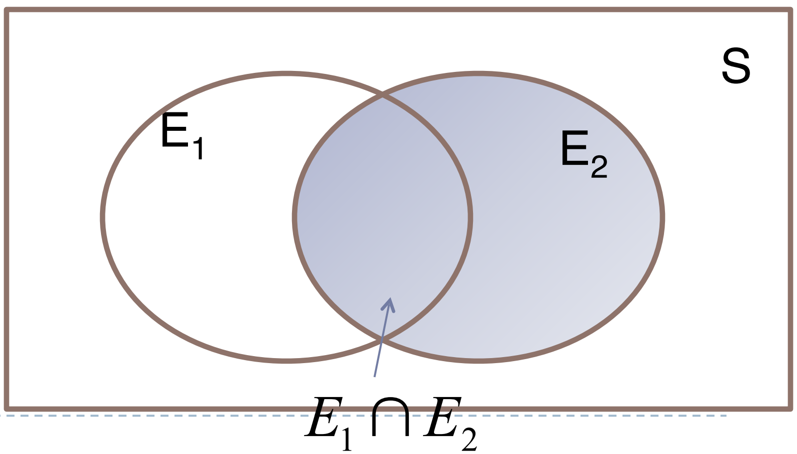
- Given E_2 occurs, our sample space is now E_2
  - Prob. that E_1 happens equals to # of outcomes in E_1 (and E_2) divided by sample space size, and hence above definition.



General Product Rule*

- Conditional probability $\Pr(E_1 \mid E_2) = \dfrac{\Pr(E_1 \cap E_2)}{\Pr(E_2)}$ leads to general product rule:
- If E_1 and E_2 are any events in a given experiment, the probability that both E_1 and E_2 occur is given by

$$P(E_1 \cap E_2) = P(E_2) * P(E_1 \mid E_2)$$
$$= P(E_1) * P(E_2 \mid E_1).$$



Using product rule

- Two marbles are chosen from a bag of 3 red, 5 white, and 8 green marbles, without replacement
  - What's the probability that both are red ?
    - Pr(first one is red and second one is red) =?
    - Pr (First one is red)=3/16
    - Pr (second one is red | first one is red) = 2/15
    - Pr (first one is red and second one is red)

    = Pr(first one is red) * Pr(second one is red | first one is red)

    = 3/16*2/15

Using product rule

- Two marbles are chosen from a bag of 3 red, 5 white, and 8 green marbles, without replacement
  - What's the probability that one is white and one is green ?
    - Either the first is white, and second is green
      - (5/16)*(8/15)
    - Or the first is green, and second is white
      - (8/16)*(5/15)
    - So answer is (5/16)*(8/15)+ (8/16)*(5/15)

Probability: outline

- Introduction
  - Experiment, event, sample space
  - Probability of events
- Calculate Probability through counting
- Sum rule and general sum rule
- Product rule and general product rule
  - Conditional probability
- Probability distribution function*
- Bernoulli process

Probability Distribution*

- How to handle a biased coin ?
  - e.g. getting head is 3 times more likely than getting tail.
- Sample space is still {H, T}, but outcomes H and T are not equally likely.
  - Pr(getting head)+Pr (getting tail) = 1
  - Pr (getting head)=3* Pr (getting tail)
  - So we let Pr(getting head)=3/4
    Pr (getting tail)=1/4

This is called a probability distribution

Probability Distribution*

- A discrete probability function, p(x), is a function that satisfies the following properties. The probability that x can take a specific value is p(x).
  1. p(x) is non-negative for all real x.
  2. The sum of p(x) over all possible values of x is 1, that is
  3. One consequence of properties 1 and 2 is:

     $0 \leq p(x) \leq 1$.

Bernoulli Trials*

- **Bernoulli trial**: an experiment whose outcome is random and can be either of two possible outcomes
  - Toss a coin: {H, T}
  - Gender of a new born: {Girl, Boy}
  - Guess a number: {Right, Wrong}
  - ….

Bernoulli Process*

- Consists of repeatedly performing independent but identical Bernoulli trials
- Example: Tossing a coin five times
  - what is the probability of getting exactly three heads?
  - What's the probability of getting the first head in the fourth toss ?

Conditional probability, $\Pr(E_1 | E_2)$

So far we see example where E_1 naturally depends on E_2.

We next see a different example.

Calculating conditional probability*

- Toss a fair coin twice, what's the probability of getting two heads (E_1)given that at least one of the tosses results in heads (E_2) ?
  - First approach: guess ?

Conditional Prob. Example*

- Toss a fair coin twice, what's the probability of getting two heads (E_1) given that at least one of the tosses results in heads (E_2) ?
  - Second approach
    - Given that at least one result is head, our sample space is {HH,HT,TH}
    - Among them event of interest is {HH}
    - So prob. of getting two heads given … is 1/3

$$P(E_1 \mid E_2) = \frac{|E_1 \cap E_2|}{|E_2|} = \frac{1}{3}$$

Conditional Prob. Example*

- Toss a fair coin twice, what's the probability of getting two heads (E_1) given that at least one of the tosses results in heads (E_2) ?
  - Third approach

$$P(E_1 \mid E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)} = \frac{P(E_1)}{P(E_2)} = \frac{1/4}{3/4} = 1/3$$

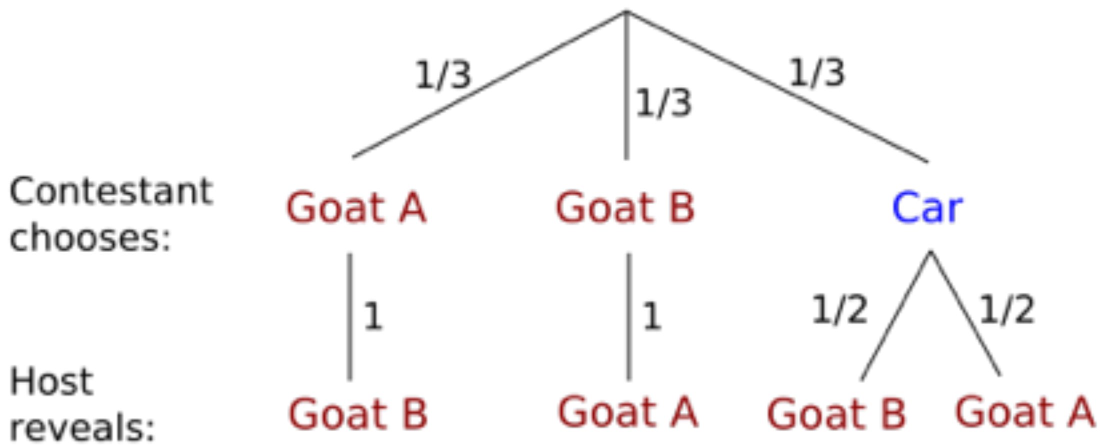
Example 2*

- In a blackjack deal (first card face-down, second card face-up)
  - T: face-down card has a value of 10
  - A: face-up card is an ace
  - Calculate P(T|A)
    - Pr(T|A)=4/51
  - Use P(T|A) to calculate P(T and A)
    - P(T and A) = Pr(A)*Pr(T|A)=4/52*4/51
  - Use P(A|T) to calculate P(T and A)
    - P(T and A)=Pr(T)*Pr(A|T)=4/52*4/51

Monty Hall Problem***

- You are presented with three doors (door 1, door 2, door 3). one door has a car behind it. the other two have goats behind them.
- You pick one door and announce it.
- Monty counters by showing you one of the doors with a goat behind it and asks you if you would like to keep the door you chose, or switch to the other unknown door.
- Should you switch?

Monty Hall Problem***



	1/3	1/3	1/6	1/6
Stay:	Goat A	Goat B	Car	Car
Switch:	Car	Car	Goat A	Goat B