Sequence

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Outline

- Sequence: finding patterns
- Math notations
  - Closed formula
  - Recursive formula
- Two special types of sequences
- Conversion between closed formula and recursive formula
- Summations

Let's play a game

What number comes next?

1, 2, 3, 4, 5, _____ 6

2, 6, 10, 14, 18, _____ 22

1, 2, 4, 8, 16, _____ 32

What comes next?

2, 5, 10, 17, 26, 37, _____ 50

1, 2, 6, 24, 120, _____ 720

2, 3, 5, 8, 12, _____ 17

1, 1, 2, 3, 5, 8, 13, _____ 21

The key to any sequence is to discover its pattern

- The pattern could be that each term is somehow related to previous terms
- The pattern could be described by its relationship to its position in the sequence (1st, 2nd, 3rd etc…)
- You might recognize the pattern as some well known sequence of integers (like the evens, or multiples of 10).
- You might be able to do all three of these ways!

2, 4, 6, 8, 10 …

- Can we relate an term to previous terms ?
  - Second term is 2 more than the first term
  - Third term is 2 more than the second term.
  - …
  - In fact, each subsequent term is just two more than the previous one.

2, 4, 6, 8, 10 …

- Can we describe each item in relation to its position in the sequence?
  - The term at position 1 is 2
  - The term at position 2 is 4
  - The term at position 3 is 6
  - …
  - The term at position n is 2 * n

2, 4, 6, 8, 10 …

- We have found two ways to describe the sequence
  - each subsequent term is two more than the previous one
  - the term at position n is 2 * n
  - It's also the sequence of all even numbers…
- To simplify our description of sequence, mathematicians introduce notations.

Mathematical Notation

- To refer to a term in a sequence, we use lower case letters (a, b, …) followed by a subscript indicating its position in the sequence
- Ex: 2, 4, 6, 8, 10 …
  - a_1 =2 first term in a sequence
  - a_2 =4 second term in a sequence
  - a_n n-th term in a sequence , n can be any positive integers
  - a_{n+1} (n+1)-th term in a sequence

2, 4, 6, 8, 10 …

- What is a_1?
- What is a_3?
- What is a_5?
- What is a_n if $n = 4$?
- What is a_{n-1} if $n = 4$?

Recursive formula

- A **recursive formula** for a sequence is one where each term is described in relation to its previous term (or terms)
- For example:

$$\begin{cases} a_1 = 1 & \text{initial conditions} \\ a_n = 2a_{n-1} & \text{recursive relation} \end{cases}$$

- a_4=?

Fibonacci sequence

- 0, 1, 1, 2, 3, 5, 8, 13, …

$$\begin{cases} a_1 = 0 \\ a_2 = 1 \\ a_n = a_{n-1} + a_{n-2} \end{cases}$$

- What's a_{10} ?
  - Starting from a_1, a_2, …, until we get a_{10}

Fibonacci in nature

- Suppose at 1st month, a newly-born pair of rabbits, one male, one female, are put in a field.
- Rabbits start to mate when one month old: at the end of its second month, a female produce another pair of rabbits (one male, one female)
  - i.e., 2 pair of rabbits at 2nd month
- Suppose our rabbits **never die**
- Fibonacci asked: how many pairs will there be in 10th month, 20th month?

Recursion*

- Recursive formula has a correspondence in programming language: recursive function calls:

$$\begin{cases} a_1 = 0 \\ a_2 = 1 \\ a_n = a_{n-1} + a_{n-2} \end{cases}$$

- Pseudo-code for function a(n)
  - int a(n)
  - {
    - If n==1, return 0;
    - If n==2, return 1
    - Return (a(n-1)+a(n-2));
  - }

Exercises: find out recursive formula

- 1, 4, 7,10,13, …
- 1, 2, 4, 8, 16, 32, …
- 1, 1, 2, 3, 5, 8, 13, …

Closed formula

- A **closed formula** for a sequence is a formula where each term is described only by an expression only involves its position.
- Examples:
  - Can you write out the first few terms of a sequence described by $a_n = 2n$?
    - Just plug in n=1, 2, 3, … into the formula to calculate a_1, a_2, a_3, …
  - Other examples:
  -

$$c_n = n^2 \qquad b_n = 3n - 2$$

To find closed formula

2, 4, 6, 8, 10 …

Write each term in relation to its position (as a closed formula)

- $a_1 = 1 * 2$
- $a_3 = 3 * 2$
- $a_5 = 5 * 2$
- More generally, $a_n = n * 2$
  - The n-th term of the sequence equals to 2n.

Exercises: find closed formula

- 1, 3, 5, 7, 9, …

- 3, 6, 9, 12, …

- 1, 4, 7, 10, 13, …

Closed formula vs. recursive formula

- Recursive formula
  - Given the sequence, easier to find recursive formula
  - Harder for evaluating a given term
- Closed formula
  - Given the sequence, harder to find closed formula
  - Easier for evaluating a given term

Two kinds of sequences:
* with constant increment
* exponential sequence

2, 4, 6, 8, 10 …

- Recursive formula:
  - $a_1=2$
  - $a_n=a_{n-1}+2$
- Closed formula: $a_n= 2n$

1, 4, 7, 10, 13, 16…

- Recursive formula:
  - $a_1=1$
  - $a_n=a_{n-1}+3$
- Closed formula: $a_n= 3n-2$

Any commonalities between them ?

Sequence with equal increments

- Recursive formula:

$$\begin{cases} x_1 = a \\ x_n = x_{n-1} + b \end{cases}$$

- Closed formula: $x_n = ?$

$x_2 = x_1 + b = a + b$

$x_3 = x_2 + b = (a+b) + b = a + 2b$

$x_4 = x_3 + b = a + 3b$

…

$x_n = a + (n-1)b$

Now try your hand at these.

2, 6, 10, 14, 18, ____

Recursive Formula

$$b_1 = 2$$

$$b_n = b_{n-1} + 4$$

Closed Formula

$$b_n = 4n - 2$$

Exponential Sequence

1, 2, 4, 8, 16, ____

Recursive Formula

$$\begin{cases} c_1 = 1 \\ c_n = 2c_{n-1} \end{cases}$$

Closed Formula: C_n=?

$$c_1 = 1$$

$$c_2 = 2 * c_1 = 2$$

$$c_3 = 2 * c_2 = 2 * 2$$

$$c_4 = 2 * c_3 = 2 * 2 * 2$$

$$c_n = 2^{(n-1)}$$

General Exponential Sequence

Recursive Formula

$$\begin{cases} c_1 = a \\ c_n = bc_{n-1} \end{cases}$$

Closed Formula: C_n=?

$$c_1 = a$$

$$c_2 = b * c_1 = b * a$$

$$c_3 = b * c_2 = b * b * a = b^2 * a$$

$$c_4 = b * c_3 = b * b^2 * a = b^3 * a$$

$$c_n = b^{n-1} * a$$

Exponential Sequence: example 2

1, 3, 9, 27, 81, _____

Recursive Formula

$$c_1 = 1$$

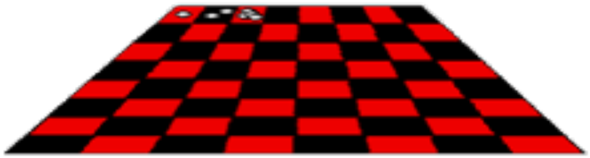
$$c_n = 3c_{n-1}$$

Closed Formula

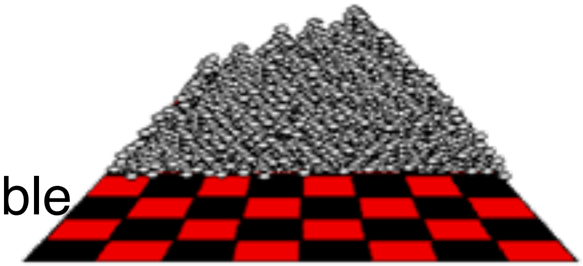
$$c_n = 3^{(n-1)}$$

A fable about exponential sequence

- An India king wants to thank a man for inventing chess
- The wise man's choice
  - 1 grain of rice on the first square
  - 2 grain of rice on the second square
  - Each time, double the amount of rice



- Total amount of rice?
  - About 36.89 cubic kilometers
  - 80 times what would be produced in one harvest, at modern yields, if all of Earth's arable land could be devoted to rice
  - As reference, Manhantan Island is 58.8 square kilometers.



Summations

Common Mathematical Notion

- Summation: A summation is just the sum of some terms in a sequence.
- For example
  - 1+2+3+4+5+6 is the summation of first 6 terms of sequence: 1, 2, 3, 4, 5, 6, 7, ….
  - 1+4+9+16+25 is the summation of the first 5 terms of sequence 1, 4, 9, 16, 25, 49, …

Summation is a very common Idea

- Because it is so common, mathematicians have developed a shorthand to represent summations (some people call this sigma notation)

$$\sum_{n=1}^{7}(2n+1)$$

This is what the shorthand looks like, on the next few slides we will dissect it a bit.

Dissecting ***Sigma Notation***

$$\sum_{n=1}^{7}(2n+1)$$

The giant Sigma just means that this represents a summation

Dissecting Sigma Notation

$$\sum_{n=1}^{7}(2n+1)$$

The ***n=1*** at the bottom just states where is the sequence we want to start. If the value was 1 then we would start the sequence at the 1st position

Dissecting Sigma Notation

$$\sum_{n=1}^{7}(2n+1)$$

The ***7*** at the top just says to which element in the sequence we want to get to. In this case we want to go up through the 7-th item.

Dissecting Sigma Notation

$$\sum_{n=1}^{7}(2n+1)$$

The part to the right of the sigma is the closed formula for the sequence you want to sum over.

Dissecting Sigma Notation

$$\sum_{n=1}^{7}(2n+1)$$

So this states that we want to compute summation of 1st, 2nd, …,7th term of the sequence given by closed formula, (a_n=2n +1).

Dissecting Sigma Notation

$$\sum_{n=1}^{7}(2n+1)$$

Thus our summation is

3 +5+7 … + 15

Let's try a few. Compute the following summations

$$\sum_{i=1}^{5}(i+2) = 3+4+5+6+7=25$$

$$\sum_{i=1}^{7}(i^2+1) = 2+5+10+17+26+37+50=147$$

How would you write the following sums using sigma notation?

$$5+10+15+20+25+30+35+40 = \sum_{i=1}^{8}(5i)$$

$$1+8+27+64+125+216 = \sum_{i=1}^{6}(i^3)$$

Summary

- Sequence: finding patterns
- Recursive formula & Closed formula
- Two special types of sequences:
  - Recursive formula => closed formula*
- Summations