Sets

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Outline on sets

- Basics
  - Specify a set by enumerating all elements
  - Notations
  - Cardinality
- Venn Diagram
- Relations on sets: subset, proper subset
- Set builder notation
- Set operations

Set: an intuitive definition

- A set is just a collection of objects, these objects are called the members or elements of the set
- One can specify a set by enclosing all its elements with curly braces, separated by commas
- Examples:
  - Set $\{a, b, c, d, e, f\}$ has 6 elements: letter a, letter b, letter c, d, e, and f
  - Set $\{bob, 1, 8, clown, hat\}$ has 5 elements: bob, 1, 8, clown, hat.

A set without elements is a special set:

$$\{\ \}$$

Called the empty set, or null set, often also denoted as ϕ

Enumerating set elements

1. You don't list anything more than once

$$\{a, b, a, b, e, f\}$$

2. Order doesn't matter

The following sets are identical (same)

$$\{1,2,3\} = \{3,1,2\}$$

So what if I get tired of writing out all of these Sets?

- Just as with algebra, we give name to a set.
- Typically we use single capital letters to denote a set.
- For example:

$$A = \{a, b, c, d, e, f\}$$

Notations

- Two key symbols that we will see:

$x \in A$ means "x is an element of set A"

$x \notin A$ means "x is not an element of set A"

$$a \in \{a, b, c\}$$

$$1 \notin \{a, b, c\}$$

Cardinality

- The cardinality of a set A is the number elements in the set, denoted as |A|.
- For example:

$$A = \{a, b, c, d, e, f\}$$

$$|A| = 6$$

$$|\{\}| = 0$$

Exercises

1. What is |A|?

   3

2. What is |B|+|C|

   6

3. What is |D|+|E|-|A|?

   3+0-3=0

$$A = \{alpha, beta, gamma\}$$

$$B = \{-5, 0, 5, 10\}$$

$$C = \{\{a, \{b\}\}, \{c, d\}\}$$

$$D = \{\{\}, \{\}, 10, 11\}$$

$$E = \{\}$$

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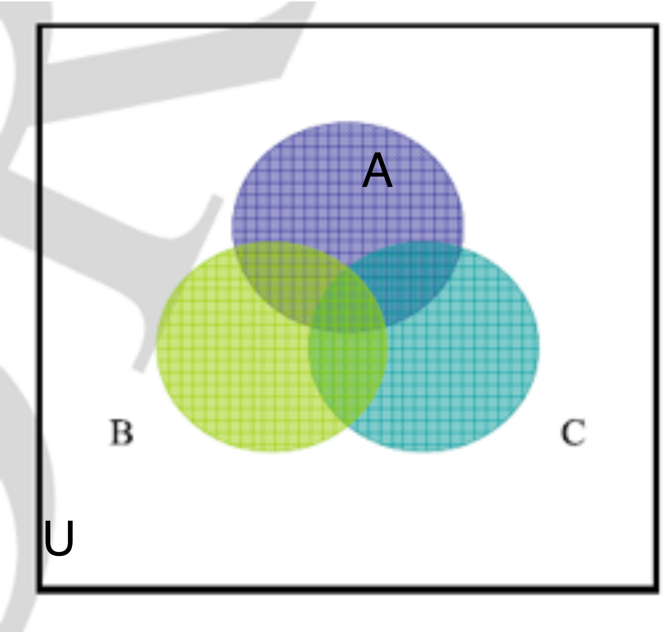
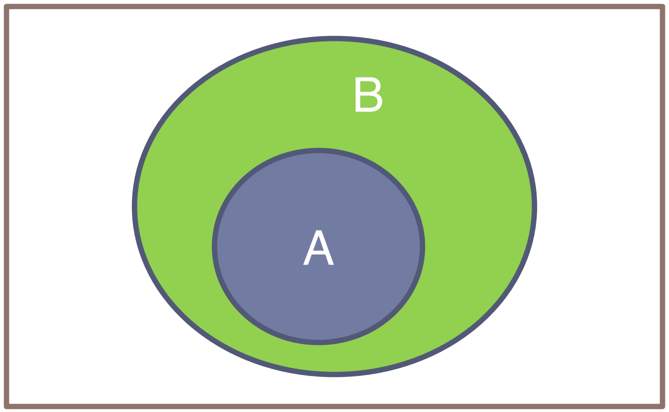
Venn Diagram

- Venn Diagram is a diagram for visualizing sets
  - a rectangle represents universal set, U, the set contains all elements that we are interested in
  - Circles within it represent other sets
- Ex: U: the set of all Fordham students
  - A: all freshman students, B: all female students,
  - C: all science major students



Relations between sets



If A is totally included in set B, i.e., every element of A is also an element of B, denoted as $A \subseteq B$, read as A is a subset of B

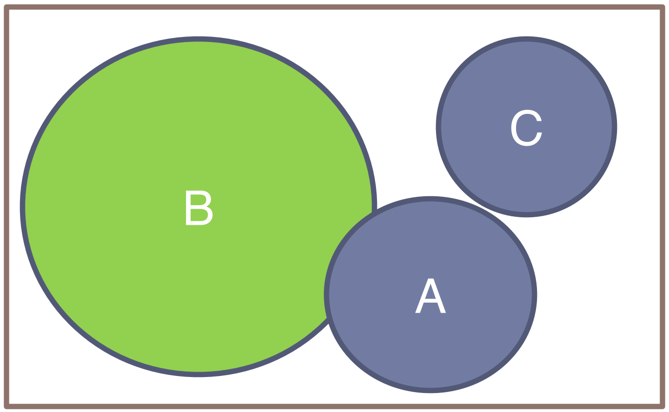
For example:

$$\{1,3,5\} \subseteq \{1,2,3,4,5\}$$

$$\{1,2,3\} \subseteq \{1,2,3\}$$ Any set is a subset of itself

$$\{\} \subseteq \{1,2,3,4,5\}$$ Empty set is subset of any set

Relations between sets



If A is not totally included in set B, i.e., there exists some element of A that is not an element of B, then A is not a subset of B, denoted as

$$A \nsubseteq B$$

For example:

$$\{1,3,6\} \nsubseteq \{1,2,3,4,5\}$$

$$\{1,2,3\} \nsubseteq \{4,5\}$$

Proper subset

- If A is a subset of B, and A≠B, then A is a proper subset of B, denoted as $A \subset B$
  - Analogy to ≤ and < relations between numbers

$$\{1,2,3,4,5\} \subseteq \{1,2,3,4,5\}$$
$$\{1,3,5\} \subset \{1,2,3,4,5\}$$

Exercise: True or False

1. If $x \in A$, and $A \subseteq B$, then $x \in B$
2. If $A \subseteq B$, and $B \subseteq C$, then $A \subseteq C$
3. If $A \subseteq B$, then $|A| \leq |B|$
4. If $|A| \leq |B|$, then $A \subseteq B$
5. {} has no subset.

Exercise

- Find out all subsets of set A={1,2}

- Find out all subsets of set A={a,b,c}

- Find out all proper subsets of set A={a,b,c}

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Some well-known sets

- **N** is the natural numbers {0, 1, 2, 3, 4, 5, …}

  $$1 \in N \quad -10 \notin N \quad 3.1415 \notin N$$

- **Z** is the set of integers {…-2,-1,0,1,2,…}
- **Q** is the set of rational numbers
  - Any number that can be written as a fraction, that is $\frac{p}{q}$, where p and q are integers, and q≠0
  - e.g. $\pi \notin Q \quad \sqrt{2} \notin Q$
- **R** is the set of real numbers
  - all numbers/fractions/decimals that you can imagine, including π, $\sqrt{2}$ etc.

Some Well-known Sets: Variations

- $\mathbf{N}^{+}$ is the set of positive natural numbers, { 1, 2, 3, 4, 5, …}
- $\mathbf{Z}^{-}$ is the set of negative integers {-1,-2,-3,…}
- $\mathbf{Q}^{>1}$ is the set of rational numbers that are greater than 1
- $\mathbf{R}^{<10}$ is the set of real numbers that are smaller than 10

Set Builder Notation

- We don't always have the ability or want to list every element in a set.
- Mathematicians have invented "Set Builder Notation". For example,

$$\{x : x \in N \text{ and } x > 10\}$$

$$\{x \mid x \in N \text{ and } 3x > 10\}$$

read as "a set contains all x's such that x is an element of the set of natural numbers and … "

Set Builder Notation

$$\{x : x \in N \text{ and } x > 10\}$$

first half: what we want to include in our set

Second half: constrains on objects specified in first half for it to be an element of the set.

Reading set builder notations

$\{x : x \times 2 = 5\}$ $\{2.5\}$

$\{x : x = 2k \text{ and } k \in \{1,2,3\}\}$ $\{2,4,6\}$

$\{x : x \in N \text{ and } \frac{x}{3} \in N\}$

$\{0,3,6,9,12,15,\ldots\}$

$\{x \mid x = 2y \text{ for some y} \in Z^{+}\}$

$\{2,4,6,8,10,12,\ldots\}$

More about set builder

- First half: can be an expression, or specify part of the constraints.
- For example:
  - Let A={1,2,3,5}

$$\{x \in A : x \text{ is even}\} = \{2\}$$

$$\{x + 4 : x \in \{1,2,3\}\} = \{5,6,7\}$$

$$\{x + y : x \in A \text{ and } y \in \{1,2,3\}\}$$

$$= \{2,3,4,5,6,7,8\}$$

Some exercise

- Find all elements of set B defined as follows:

$$B = \{2x : x \in N \text{ and } x \leq 4\}$$

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Set Operations

- Just like in arithmetic, there are lots of ways we can perform operation on sets. Most of these operations are different ways of combining two different sets, but some (like Cardinality) only apply to a single set.

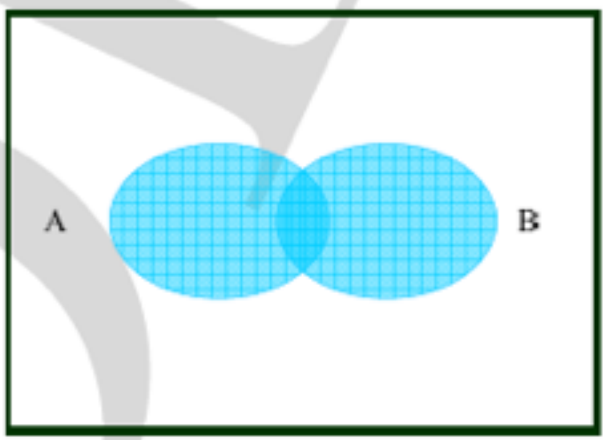
Union

$$A \cup B$$

Create a new set by combining all of the elements or two sets, i.e.,

$$A \cup B := \{x \mid x \in \mathrm{A} \text{ or } x \in B\}$$

“is defined as”



The part that has been shaded.

Union Examples

$$A = \{1,2,3,4,5\}$$

$$B = \{0,2,4,6,8\}$$

$$C = \{0,5,10,15\}$$

$$D = \{\}$$

$$A \cup B = \{0,1,2,3,4,5,6,8\}$$

$$B \cup A = \{0,1,2,3,4,5,6,8\}$$

$$C \cup D = \{0,5,10,15\}$$

$$(A \cup C) \cup (D \cup B) = \{0,1,2,3,4,5,6,8,10,15\}$$

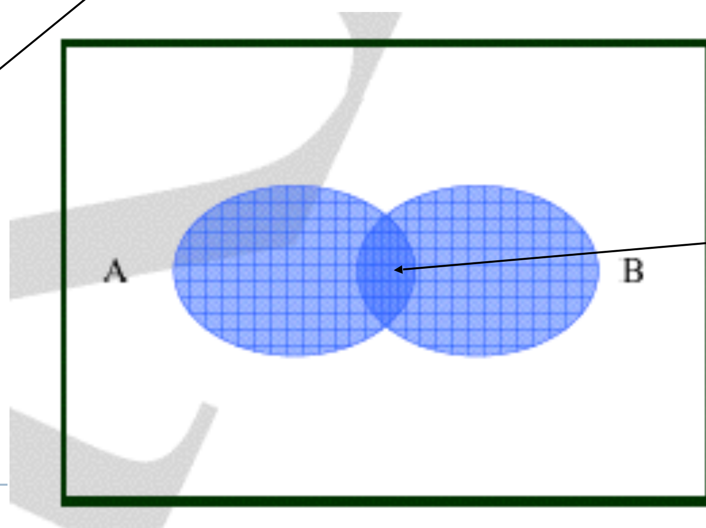
Intersection

$$A \cap B$$

Create a new set using the elements the two sets have if common

$$A \cap B := \{x \mid x \in \mathrm{A} \text{ and } x \in B\}$$

“is defined as”



The part that has been shaded twice is $A \cap B$

Intersection Examples

$$A = \{1,2,3,4,5\}$$

$$B = \{0,2,4,6,8\}$$

$$C = \{0,5,10,15\}$$

$$D = \{\}$$

$$A \cap B = \{2,4\}$$

$$B \cap A = \{2,4\}$$

$$C \cap D = \{\}$$

$$(A \cap C) \cup (D \cap B) = \{5\}$$

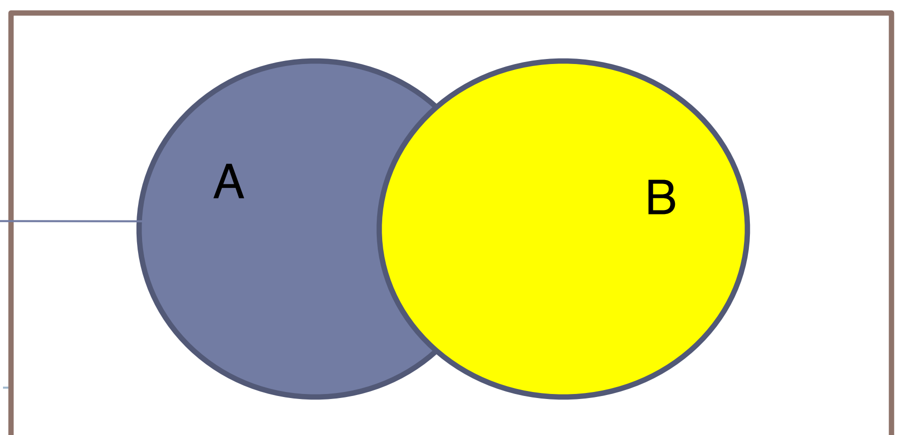
Difference

$$A - B$$

Create a new set that includes all elements of set A, removing those elements that are also in set B

$$A - B := \{x \mid x \in \mathrm{A} \text{ and } x \notin B\}$$

A-B: the part that is shaded in blue



Difference Examples

$$A = \{1,2,3,4,5\}$$

$$B = \{0,2,4,6,8\}$$

$$C = \{0,5,10,15\}$$

$$D = \{\}$$

$$A - B = \{1,3,5\}$$

$$B - A = \{0,6,8\}$$

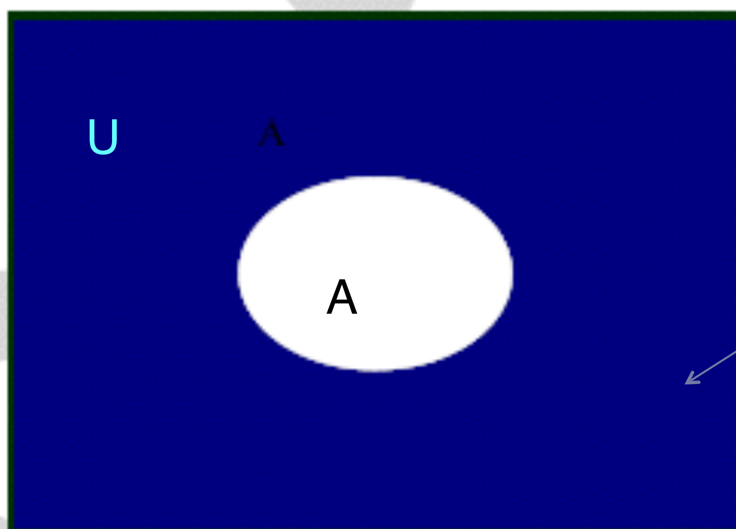
$$C - D = \{0,5,10,15\}$$

$$(A - C) - (D - B) = \{1,2,3,4\}$$

Complement

The difference of universal set U (the set that includes everything) and A is also called the complement of A:

$$A^c := U - A = \{x \mid x \in U \text{ and } x \notin A\}$$



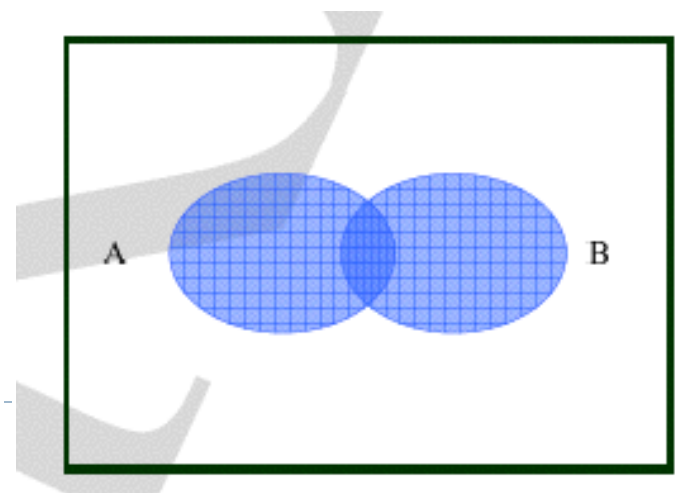
Colored area is U-A

Set operations example

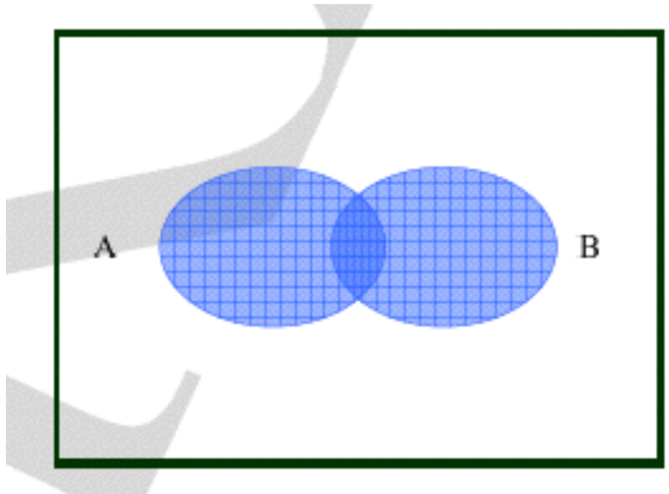
- A is the set of all computer science majors.
- B is the set of all physics majors.
- C is the set of all general science majors.
- D is the set of all female students.
- Using set operations, describe each of the following in terms of the sets A, B, C and D:
  - Set of all male physics majors.
  - Set of all students who are female or general science majors.
  - Set of all students not majoring in computer science.

Fenway Park or Yankee Stadium

- Of the 28 students in a class,
  - 23 have visited one or both
  - 12 have visited Fenway
  - 9 have visited Fenway and Yankee Stadium
- How many have visited Yankee stadium?
- How many have visited only Fenway Park?



Principle of Inclusion/Exclusion



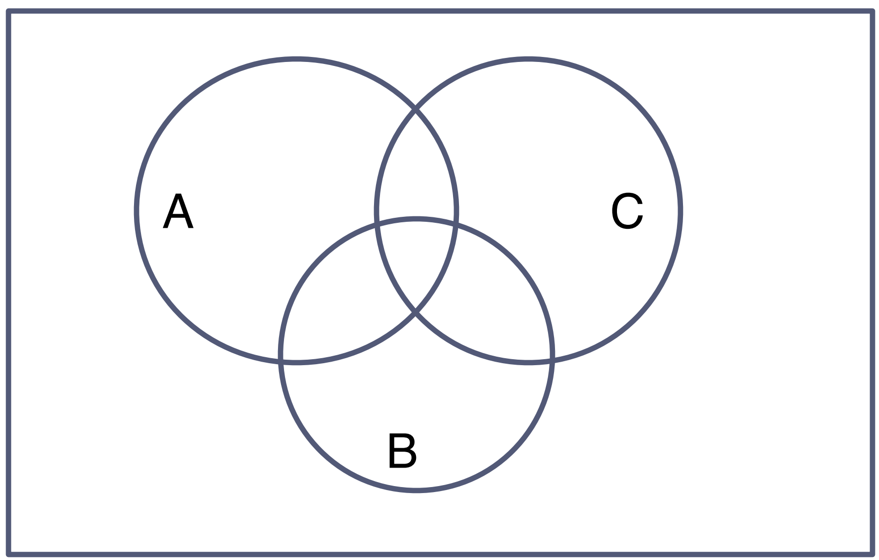
$$|A \cup B| = |A| + |B| - |A \cap B|$$

Also,

$$|A| = |A \cup B| - |B| + |A \cap B|$$

Principle of Inclusion/Exclusion

$$|A \cup B \cup C| =$$



Power Set

$$P(A)$$

is a set that consists of all subsets of set A. $P(A) := \{x : x \subseteq A\}$

e.g. P({1})=?

List all subsets of {1}: {},{1}

Therefore P({1})={{},{1}}.

Power Set Examples

$$A = \{1,2,3\}$$

$$B = \{a,b,c,d\}$$

$$P(A) = \{\,\{1,2,3\},\{1,2\},\{1,3\},\{2,3\},\{1\},\{2\},\{3\},\{\}\,\}$$

$$P(B) = \{\,\{a,b,c,d\},\{a,b,c\},\{a,b,d\},\{a,c,d\},\{b,c,d\},\{a,b\},\{a,c\},\{a,d\},\{b,c\},\{b,d\},\{c,d\},\{a\},\{b\},\{c\},\{d\},\{\}\,\}$$

Exercises on power set

1. $A = \{\}$

$P(A) =$

2. $\mathrm{C} = \{\mathrm{a}, 1, \{\}\}$

$\mathrm{P(C)} =$

Cardinality of Power Set

- If |A|=1, |P(A)|= ?
  - Try P({a})=
- If |A|=2, |P(A)|=
  - Try P({a,b})
- If set A has a certain number of subsets, after we add one more element into A, how many subsets A has now ?
  - Every originally identified subsets are still valid
  - Add the new element into each of them, and we get a new subset.
  - The number of subsets doubles !

Cardinality of Power Set

- Set A with n elements has a_n subsets

$$a_1 = 2$$
$$a_n = 2a_{n-1}$$

- We can find the closed form:

$$a_n = 2^n$$

- A set of cardinality n has 2^n subsets
- If $|A|=n$, $|P(A)|=2^n$
- Or: $|P(A)|=2^{|A|}$

Cartesian Product (Cross Product)

$$A \times B$$

Create a new set consisting of all possible ordered pairs with the first element taking from A, and second element taking from B.

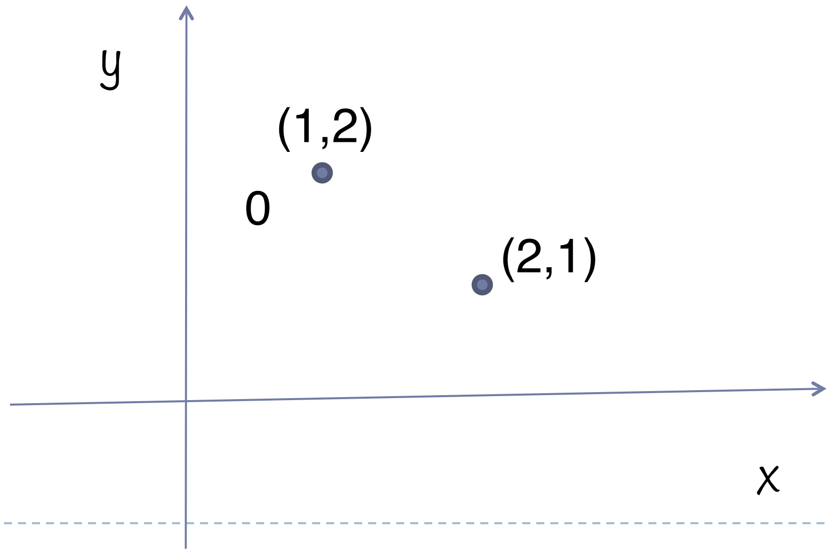
$$A \times B := \{(x, y) : x \in A \text{ and } y \in B\}$$

Ordered Pair (x, y)

- Its just like what you learned about when you learned about graphing:

  $$(1,2) \neq (2,1)$$

- It's different from set !
  - {1,2}={2,1}
- x, and y can be numbers, names, anything you can imagine



Example of Cartesian Product

- I have two T-shirts: white, black
  - A={white shirt, black shirt}
- I have three jeans: black, blue, green
  - B={black jean, blue jean, green jean}
- All outfits I can make out of these ?
  - The set of all ordered-pairs, in the form (T-shirt, jean)…

Cartesian Product Examples

$$A = \{1,2,3\}$$

$$B = \{a,b,c\}$$

$$A \times B = \{(1,a),(1,b),(1,c),(2,a),(2,b),(2,c),(3,a),(3,b),(3,c)\}$$

$$B \times A = \{(a,1),(a,2),(a,3),(b,1),(b,2),(b,3),(c,1),(c,2),(c,3)\}$$

Cardinality of Cartesian Product

- If A has m elements, B has n elements, how many elements does AxB have ?
  - For every element of A, we pair it with each of the n elements in B,. to get n ordered pairs in AxB
  - So we can form n*m ordered pairs this way
  - So |AxB| = m*n = |A|*|B|
- This is where the name Cartesian product comes from.

Exercises on Power Set/Cartesian Product

- {} x {1,2}=
- P({a,b}) x {c,d} =
- Is it true that for any set A, $\{\} \in P(A)$?
- Is it true that for any set A, $\{\} \subseteq P(A)$?

Examples

- A certain club is forming a recruitment committee consisting of five of its members. They have calculated that there are 8,568 different ways to form such a committee. The club has two members named Jack and Jill. They have calculated that 2,380 of the potential committees have Jack on them, 2,380 have Jill, 1,820 have Jack but not Jill, 1,820 have Jill but not Jack, and 560 have both Jack and Jill.
  1. How many committees have either Jack or Jill?
  2. How many committees have neither Jack nor Jill?
  3. Jack and Jill are car-pooling, so they insist that if either one is on the committee, the other person must also be on the committee. How many committee meet this condition?
  4. Jack and Jill have had a fight. Jack says, “if Jill is on the committee, I won’t be.” Jill says,“Likewise.” How many of the committees meet this condition?

(Hint, draw a Venn diagram depicting the set of all possible committees as the universal set…)

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