CISC 4090
Theory of Computation

Non-regular languages

Professor Daniel Leeds
dleeds@fordham.edu
JMH 332

Regular languages

Definition: a language is called a regular language if some finite automaton recognizes it

What languages cannot be recognized by an FSA

Regular languages use finite memory (finite states)

Non-regular languages require infinite memory

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Are the following regular?

L1 = {w | w has at least 100 1's}

Yes: Start at q_0, For each 1 q_k -> q_{k+1} . F={q_{100}}

L2 = {w | w has same number of 0's and 1's}

No: unknown number of states

L3 = {w | w is of the form 0^n1^n, n>0}

No: unknown number of states

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What about this class of languages

$\Sigma = \{a, b\}$

L_n={w | w contains n b's in a row }

- L_3={abbba, aabbba,ababbbba, ...}
- L_4={babbbbab, bbbb, aaabbbbab, ...}

L_n is regular for each value of n

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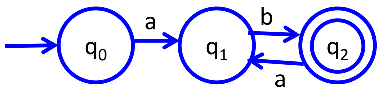
Regular languages can be infinite

- E.g., $a(ba)^*b$



For FSA to generate an infinite set of strings, there must be **a loop** between some states

Pumping lemma

Every string in regular language L with length greater than or equal to the pumping length *p* can be "pumped"

Every string $s \in L$ ($|s|>=p$) can be written as xyz where

1. For each $i \geq 0$, $xy^iz \in L$
2. $|y| > 0$
3. $|xy| \leq p$

If L violates pumping lemma,
then it is not regular

Pumping lemma, continued

1. For each $i \geq 0$, $xy^iz \in L$
   There is a loop
2. $|y| > 0$
   There is a loop of letters (not of ε, which would effectively not be a loop)
3. $|xy| \leq p$
   Not allowed more states than pumping length (keep memory finite!)

Example: $L=(01)^*$

w=10101010 ∈ L

Can divide into: x=1010, y=10, z=10

xy^2z = 1010 1010 10 -> 1010101010 ∈ L

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w=10 ∈ L

$x=\varepsilon$ y=10 $z=\varepsilon$

$xy^2z = \varepsilon$ 1010 ε -> 1010 ∈ L

$xy^0z = \varepsilon\ \varepsilon\ \varepsilon \rightarrow \varepsilon \in$ L

Proof idea

If $|s| \leq p$, trivially true

If $|s|>p$, consider the states the FSA goes through

- Since there are only p states, $|s|>p$, one state must be repeated
- **Pigeonhole principle:** There must be a cycle

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Common pumping proof-by-contradiction

Define a simple word w that is guaranteed to have more than p symbols, and you know the first p symbols

Show repetition of intermediate y string violates language rules

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Prove $B=\{0^n1^n\}$ is not regular

B={01, 0011, 000111, 00001111, ….}

Proof by contradiction: assume B is regular

thus, any $w \in B$ can be "pumped" if $|w|>p$

First suggestion: w=0011, x=0, y=01, z=1 – counterexample

$xy^2z=001011 \notin B$

Close! But maybe $|0011| \leq p$, how do we know this will be problem when $|w|>p$

Our solution: Let $\mathbf{w=0^p1^p}$ $|w|>p$, so **must** be "pump"-able

$|xy| \leq p$ so, $x=0^f$ $y=0^g$, $f+g \leq p$ and $g>0$

When we pump w: xy^2z,

we get $p+g$ 0's followed by p 1s. $xy^2z \notin B$

Contradiction, pumped w $\notin B$

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Prove $F=\{ww \mid w=(0 \cup 1)^* \}$ is not regular

F={11, 00, 0101, 1010, 11011101, …}

Proof by contradiction:

assume F is regular

thus, any $v \in F$ can be "pumped" if $|v|>p$

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Prove F={ww | w=$(0 \cup 1)^*$ } is not regular

Proof by contraction: assume F is regular

thus, any $v \in F$ can be "pumped" if |v|>p

- Our solution: Let w=0^p10^p1 |w|>p so must be "pump"-able

  $|xy| \leq p$ so, x=0^f y=0^g, $f+g \leq p$ and g>0

  When we pump w: xy^2z,

  we get *p+g* 0's followed by 10^p1 . $xy^2z \notin B$

  Contradiction, pumped w $\notin F$

F={11, 00, 0101, 1010, 11011101, …}

Prove E={1^{n^2}} is not regular

Proof by contradiction: assume E is regular

thus, any $w \in E$ can be "pumped" if |w|>p

Our solution: Let $\mathbf{w = 1^{p^2}}$ |w|>p, so **must** be "pump"-able

$|xy| \leq p$ so $|y| \leq p$

$|xy^2z| \leq p^2 + p$

What's the length of the next-biggest string after $|w|=p^2$

$|w^{next\text{-}biggest}| = (p+1)^2 = p^2+2p+1$

Pumping w once gives length at most $p^2+p < p^2+2p+1$

Thus, $xy^2z \notin E$

Contradiction, pumped w $\notin E$

Prove A={0^i1^j | i>j>0} is not regular

Proof by contradiction: assume A is regular

thus, any $w \in A$ can be "pumped" if |w|>p

Prove A={0^i1^j | i>j>0} is not regular

Proof by contradiction: assume A is regular

thus, any $w \in A$ can be "pumped" if |w|>p

Our solution: Let **w=$0^{p+1}1^p$** |w|>p, so **must** be "pump"-able

$|xy| \leq p$ so, x=0^f y=0^g, $f+g \leq p$ and g>0

Let's say xy = 0^p So z=01^p

When we pump w: xy^2z,
we get $0^f0^g0^g01^p$ -> $0^{p+g+1}1^p \in A$

Let's try pumping **down**: xy^0z,
we get xz -> 0^f01^p
Number of 0s: f+1 Number of 1s: p=f+g≥f+1
f+1≤p number of 0s<number of 1 $xy^0z \notin A$

Contradiction, pumped w $\notin A$