CISC 4090
Theory of Computation

Decidability

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JMH 332

"Turing recognizable" vs. "Decidable"

Language is **Turing recognizable** if some Turing machine recognizes it

- Also called "recursively enumerable"

Machine that halts on all inputs is a **decider**. A decider that recognizes language L is said to **decide** language L

Language is **Turing decidable,** or just **decidable**, if some Turing machine decides it

Not all problems can be solved

- Good to know when you might not find an answer
- Get perspective on limits of computation

Decidable problems for regular languages

- Does DFA D accept string s?
- Is L(D) of DFA empty?
- Are two DFAs D1 and D2 equivalent?

Specify DFA on input TM,
determine control algorithm to run DFA specified on tape

Arbitrary DFA D accepts string w

Language: A_{DFA}={(D,w) | D is DFA that accepts w}

Theorem: A_{DFA} is decidable

Proof idea:

~StartQ#AcceptQ#δ#CurrentState#w~

- Define machine M that simulates D on w
- If simulation ends in an accept, accept; else, reject

Note: control states in M cannot be states in D

M needs to run arbitrary D

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A_{DFA} decider Proof Outline

DFA D described as string: 5-tuple

Use marks on tape to track

- current state in simulated D
- current symbol read from w

Implement transition function of D for current D state and input w

- D's transition δ is **different** from TM M's transition δ

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Arbitrary DFA D accepts no strings

E_{DFA}={ D | D is DFA with L(D)={} } is decidable language

Proof idea:

- Is there any way to reach accept from start?
- Think of graph-marking

Proof

- Mark start state of DFA D
- Repeat until no new states
  - Mark any state that past-marked states transition to
- If an accept state is marked, REJECT; else, accept

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Two DFAs are equivalent

EQ_{DFA} = {(A,B) | A and B are DFAs and L(A)=L(B)} is decidable language

Proof idea:

- Construct new DFA C from A and B; C accepts only strings accepted by either A or B, but not both
- Check if C's language is empty (last slide)

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Two DFAs are equivalent

EQ_{DFA} = {(A,B) | A and B are DFAs and L(A)=L(B)} is decidable language

Alternative proposals:

- Proposal 1: For machine A, start at start state and record path from start to all possible accept states; do the same for B; check if bijection between unique A and B paths – might work, but need to confirm how this will deal with self loops
- Proposal 2: Create enumerator for machine A. Loop on i=1 to infinite, print out strings in L(A) with length i and test if B accepts – theoretically works, but takes infinite time for infinite words – will never halt (this solution does not show decidability)

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A_{CFG} is decidable – Proof

For CFG G and string w, determine if G generates w

Idea 1: Simulate G to go through all derivations

- May never terminate

Idea 2: Note |w|=n; 2n-1 steps from CNF rules to each string
Produce all words of lengths n

- Breadth-first search of finite depth is fixed

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B_{CFG} is a decidable language

- For CFG G, determine if there is any terminal string generated by G
- Mark all variables that generate terminals
- Repeated loop:
- Mark all variables that have previously-marked variables on its rules right sides
- If mark S, ACCEPT; otherwise reject

S -> AB
A -> An | x
B -> yB | d

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B_{CFG} is a decidable language

- For CFG G, determine if there is any terminal string generated by G

Alternative solution:

- Breadth-first search of rules; block repeat-visits to a branch
- If you reach an all-terminal solution, accept
- If never reach all-terminal solution, reject

Are there a finite number of non-breadth-first traversals? **Yes**

Rule can lead to *k* possible variables: S-> ABACCD…E
Each of *m* variables in grammar can have *p* possible rules
Given no loops allowed, max # rule evals per variable in first rule is *pm*
Max bound on number of steps: $p\ (pm)^k$ … huge but finite

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EQ_{CFG} is not a decidable language

- Regular expressions closed under complement and intersection
- CFLs not closed under complement and intersection
- We will prove non-decidable languages later

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The Halting Problem

Key theorem to theory of computation

Addressing unsolvable problems

Unsolvable: Software verification

- For arbitrary computer program P and precise specification of program's behavior S, determine if P fulfills S

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Halting Problem specified

A_{TM} = {(M,w) | M is a TM and M accepts w}

- If M loops forever on w, our TM for A_{TM} must reject w
- This problem is Turing recognizable, but not decidable!

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Detour: Cantor diagonalization

Comparing sizes of two infinite sets

- What is larger: set of even positive integers or set of all strings in (0U1)*

Diagonalization: two sets have same size if each element of set A can be compared with one element of set B

From CISC 1400: Can you define bijection from set A to set B?

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Example pairing

N = {1,2,3,4,…} and E={2,4,6,8,…}

- N and E have "same size" because there exists bijection from N to E
- f(x)=2x

Set is **countable** if either it is finite or if it has same size as N

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Q is countable

Let Q = {m/n: m,n ∈ N}, positive rational numbers

Follow diagonal, skipping redundant values

1/1	1/2	1/3	1/4	1/5	1/6
2/1	2/2	2/3	2/4	2/5	2/6
3/1	3/2	3/3	3/4	3/5	3/6
4/1	4/2	4/3	4/4	4/5	4/6
5/1	5/2	5/3	5/4	5/5	5/6
6/1	6/2	6/3	6/4	6/5	6/6

Concatenating infinite set of finite lists produces countable list

Take countable steps along diagonal line to reach each number in Q

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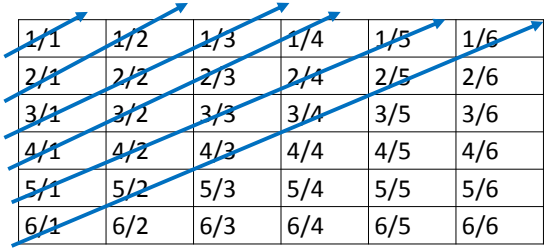

Concatenating infinite set of finite lists produces countable list

Take countable steps along diagonal line to reach each number in Q

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Real numbers are uncountable

Real numbers have infinite number of decimal places

Proving uncountability

- Presume we have a list of *n* real numbers
- Generate new real number x **not** in current list
  - Pick i^{th} decimal value of x to be different from i^{th} decimal value of element i in list of real numbers
- At end, x will not be in list

R(1)	1.532532
R(2)	0.352144
R(3)	5.244525
R(4)	9.327431
R(5)	5.366324
R(6)	4.459322
⋮	⋮
x	3.646311

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Uncountability implications

There are uncountably many languages

There are countably many Turing machines

Some languages are not Turing recognizable

Are these statements true?

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"There are countably many Turing machines"

Each TM is captured by finite string <M>∈ Σ^*

- Σ^* is countable – add number of strings of length 0, length 1, length 2, … (like Q)

"There are uncountably many languages"

Represent L as binary sequence

- 1 for each accepted string, 0 for each reject string
- Infinite number of strings – infinite sequence of 0/1s
- Set of possible binary sequences is uncountable (like R)

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"There are uncountably many languages"

Represent L as binary sequence

- 1 for each accepted string, 0 for each reject string
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"Some languages are not Turing decidable"

Set of TMs is countable

Set of Languages is uncountable

Each TM has one language

Some languages not recognized by any TM

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Back to the Halting Problem

A_{TM}={<M,w> | M is a TM and accepts w}

- Proof by contradiction **– uses diagonalization**

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Contradiction

Assume A_{TM} is decidable

H decides A_{TM}

- Input <M,w> causes H to accept if M accepts w, otherwise H rejects

Define a TM D that calls H on <M,<M>>, then outputs opposite answer to H

- D rejects if M accepts <M>; D accepts if M does not accept <M>

Run D on itself

- D(<D>) = accept if D does not accept <D>; reject if D accepts <D>

Contradiction!

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Diagonalization

	<M1>	<M2>	<M3>	<M4>	…	<D>
M1	Acc	Rej	Rej	Acc	…	Acc
M2	Rej	Rej	Acc	Rej	…	Rej
M3	Acc	Rej	Acc	Acc	…	Acc
M4	Rej	Acc	Rej	Acc	…	Rej
⋮	⋮	⋮	⋮	⋮		
D	Rej	Acc	Rej	Rej	…	**XX**

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Implications

A_{TM}={(M,w) | M is a TM and M accepts W} is **not** decidable

Some algorithms are decidable

A_{TM} **is** Turing recognizable – just **simulate** M on machine

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Co-Turing Recognizable

Language is co-Turing recognizable if it is the complement of a Turing-recognizable language

Theorem: Language is decidable if it is Turing-recognizable and co-Turing recognizable

Thus, for any undecidable language L, either L or L' is not Turing-recognizable

- Is A_{TM}' Turing-recognizable?

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Reducibility

If A reduces to B, solution to B will solve A

Example: A: Navigate NYC B: Reading a map

If A reduces to B

- A is no harder than B
- A could be easier than B

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Reduction and decidability

If A is reducable to B and B is decidable

- A is decidable

If A is reducible to B and A is undecidable

- B must be undecidable

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$HALT_{TM}$ is undecidable

We can reduce A_{TM} (TM **accepts** w) to $HALT_{TM}$ (TM **halts** on w)

A_{TM} is undecidable, this $HALT_{TM}$ is undecidable

Proof by contradiction:

- Assume $HALT_{TM}$ is decidable – TM R
- Use R to construct TM S to decide A_{TM}
- S definition:
  - If R **does not indicate halting** for <M,w>, reject w
  - If R **does indicate halt**, simulate M on w

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