CISC 4090
Theory of Computation

Turing machines

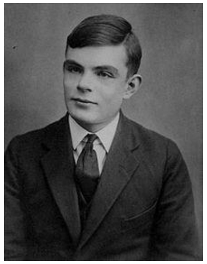
Professor Daniel Leeds
dleeds@fordham.edu
JMH 332

Alan Turing (1912-1954)

Father of Theoretical Computer Science

Key figure in Artificial Intelligence

Codebreaker for Britain in World War I

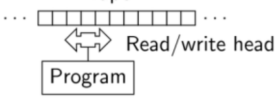


2

Turing machine

Simple theoretical machine

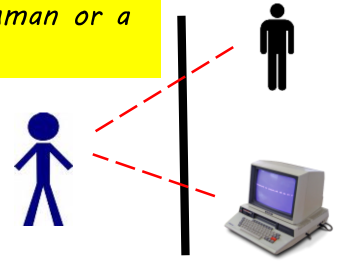
Can do anything a real computer can do!



3

Detour: "Turing test"

A computer is "intelligent" if human investigator can't tell if she's talking to a human or a computer



4

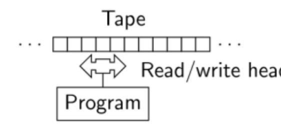
http://xkcd.com/329/

5

Turing machine

Simple theoretical machine

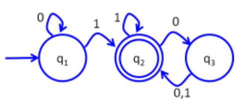
Can do anything a real computer can do!



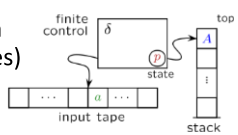
6

Review of machines

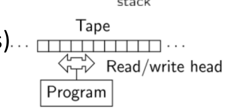
- Finite state automaton (Regular languages)



- Push down automaton (Context free languages)



- Turing machine (beyond CFLs)



7

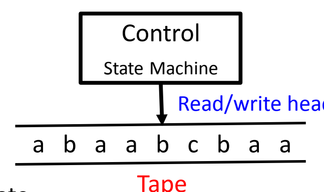
Turing machine structure

Infinite tape

At each step
- Can move left/right on tape
- Can change state

When reaches accept or reject state, terminates and outputs “accept” or “reject”

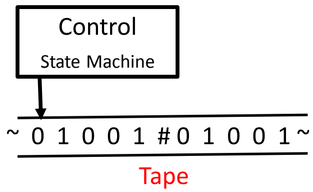
Can loop forever



8

A Turing Machine for B={w#w|$w \in \{0,1\}^*$}

Assume the string is written on the tape and you start at the beginning of the string. What can we do?



9

Strategy:

Find left-most 0-or-1 character in first word

If match left-most character in second word, X out both

Else reject

If no characters left, accept

```
~ ~ ~ 0 1 1 0 0 0 # 0 1 1 0 0 0 ~ ~ ~
~ ~ ~ X 1 1 0 0 0 # X 1 1 0 0 0 ~ ~ ~
~ ~ ~ X X 1 0 0 0 # X X 1 0 0 0 ~ ~ ~
```

How do we move this with single actions: move-by-one and write?

10

Strategy, in more detail:

Read left-most character, X it out
Move right until find #, then move right until find 0-or-1-or-~
If current character is ~ or mismatches with character before #: reject
Else, X it out
Move left until pass #, keep moving until find first X
Move one to right
If #, check right hand string,
If no extra chars, accept
If not #, go to top

Problem keeps shrinking
Will accept or reject each input

```
~ ~ ~ 0 1 1 0 0 0 # 0 1 1 0 0 0 ~ ~ ~
~ ~ ~ X 1 1 0 0 0 # 0 1 1 0 0 0 ~ ~ ~
~ ~ ~ X 1 1 0 0 0 # 0 1 1 0 0 0 ~ ~ ~
…
~ ~ ~ X 1 1 0 0 0 # 0 1 1 0 0 0 ~ ~ ~
~ ~ ~ X 1 1 0 0 0 # X 1 1 0 0 0 ~ ~ ~
```

11

Turing machine: the formal definition

7 tuple: $(Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject})$

Q is set of states

Σ is input alphabet

Γ is the tape alphabet; blank$\in \Gamma$ and $\Sigma \in \Gamma$

$\delta: Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R\}$ transition function

Start, accept, and reject state: q_0, q_{accept}, q_{reject}

12

The transition function

$\delta: Q \times \Gamma \rightarrow Q \times \Gamma \times \{L, R\}$

Given state q and symbol a at present location on tape,
change to state r, change symbol on tape to b, move Left or Right

Change in: (state, tape content, head location)
– called "configuration"

13

Some TM details for B={w#w | $w \in \{0,1\}^*$}

After X out the 0 at the far left, move right looking for the first digit after # to be 0. Use state $q_{MoveTo\#->(Then0)}$

$\delta(q_{MoveTo\#->(Then0)}, 0)$ -> $(q_{MoveTo\#->(Then0)}, 0, R)$

$\delta(q_{MoveTo\#->(Then0)}, 0)$ -> $(q_{MoveTo\#->(Then0)}, 0, R)$

$\delta(q_{MoveTo\#->(Then0)}, \#)$ -> $(q_{Find0}, 0, R)$

~ ***0*** 1 1 0 0 0 # 0 1 1 0 0 0 ~ ~
~ X ***1*** 1 0 0 0 # 0 1 1 0 0 0 ~ ~

Once we've passed #, search for matching digit for 0: q_{Find0}

$\delta(q_{Find0}, X)$ -> (q_{Find0}, X, R)

$\delta(q_{Find0}, 0)$ -> $(q_{MoveTo\#<-}, X, L)$

$\delta(q_{Find0}, 1)$ -> $(\mathbf{q_{reject}}, ?, ?)$

$\delta(q_{Find0}, \sim)$ -> $(\mathbf{q_{reject}}, ?, ?)$

14

"Turing recognizable" vs. "Decidable"

L(M) – "language **recognized** by M" is set of strings M accepts

Language is **Turing recognizable** if some Turing machine recognizes it
- Also called "recursively enumerable"

Machine that halts on all inputs is a **decider**. A decider that recognizes language L is said to **decide** language L

Language is **Turing decidable,** or just **decidable**, if some Turing machine decides it

15

Turing Machine for C={0^{2^n} | $n \geq 0$}

16

Turing Machine for C=$\{0^{2^n} \mid n \geq 0\}$

Recursive division by 2

Sweep left to right across tape, cross off every-other 0

If
- Exactly one 0: accept
- Odd number of 0s: reject
- Even number of 0s, return to front

17

Alternating 0s in action:

TM M2 "decides" language C

If you land on a location and want to cross it out, but it is a ~, you crossed out an even number of 0s – do another loop!

If you land on a location and want to skip over it, but it is a ~, you crossed out an odd number of 0s – reject!

Round 1

```
~ ~ ~ 0 0 0 0 0 0 0 0 ~ ~ ~
~ ~ ~ X 0 0 0 0 0 0 0 ~ ~ ~
~ ~ ~ X 0 0 0 0 0 0 0 ~ ~ ~
~ ~ ~ X 0 X 0 0 0 0 0 ~ ~ ~
…
~ ~ ~ X 0 X 0 X 0 X 0 ~ ~ ~
~ ~ ~ X 0 X 0 X 0 X 0 ~ ~ ~
~ ~ ~ X 0 X 0 X 0 X 0 ~ ~ ~
```

Round 2

```
…
~ ~ ~ X 0 X 0 X 0 X 0 ~ ~ ~
~ ~ ~ X 0 X 0 X 0 X 0 ~ ~ ~
…
~ ~ ~ X X X 0 X X X 0 ~ ~ ~
```

18

Language D={$a^ib^jc^k$ | k=ixj and i,j,k>0}

Multiplication on a Turing Machine!

Consider 2x3=6

```
~ ~ ~ a a b b b c c c c c c~ ~ ~
```

19

TM M3 to decide D={$a^ib^jc^k$ | k=ixj and i,j,k>0}

Scan string to confirm form is $a^+b^+c^+$
- if so: go back to front; if not: reject

X out first a, for each b, x off that b and x off one c
- If run out of c's but b's left: reject

Restore crossed out b's, repeat b—c loop for next a
- If all a's gone, check if any c's left
  - If c's left: reject; if no c's left: accept

20

"Multiply" in action:

TM M3 "decides" language D

Symbol X is an a or c that is gone for good

Symbol y is a b temporarily out of service as you go through all the other b's

Confirm $a^+b^+c^+$

~ ~ ***a*** a b b b c c c c c c ~ ~
~ ~ a ***a*** b b b c c c c c c ~ ~
…
~ ~ a a b b b c c c c c ***c*** ~ ~
…

(a,b) pair one

~ ~ ***a*** a b b b c c c c c c ~ ~
~ ~ X ***a*** b b b c c c c c c ~ ~
…
~ ~ X a y b b ***c*** c c c c c ~ ~
~ ~ X a y b ***b*** X c c c c c ~ ~
…

(a,b) pair two

~ ~ X a y y b X ***c*** c c c c ~ ~

Transducers: generating language

So far our machines accept/reject input

Transduction: Computers transform from input to output

- New TM: given *i* a's and *j* b's on tape, print out *ixj* c's

TM 4: Element distinctiveness

Given a list of strings over {0,1}, separated by #, accept if all strings are different:

Example: 01101#1011#00010

TM 4 solution

1. Place mark on top of left-most symbol. If it is blank: accept; if it is #: continue, otherwise: reject
2. Scan right to next # and place mark on it. If none encountered and reach blank: accept
3. Zig-zag to compare strings to right of each marked #
4. Move right-most marked # to the right. If no more #: move left-most # to its right and the right-most # to the right of the new first marked #. If no # available for second marked #: accept
5. Go to step 3

Decidability

How do we know decidable?

- Simplify problem at each step toward goal
- Can prove formally – number of remaining symbols at each step

Showing language is Turing recognizable but not decidable is harder

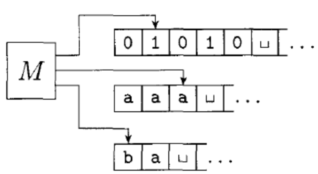
Many equivalent variants of TM

- TM that can "stay put" on tape for a given transition
- TM with multiple tapes
- TM with non-deterministic transitions

Can select convenient alternative for current problem

MultiTape TM

- Each tape has own ReadWrite Head
- Initially tape 1 has input string, all other tapes blank
- Transition does read/write on all heads at once



Equivalence of SingleTape and MultiTape TM

Convert *k* tape TM M to single tape TM S

- Contents of M's tapes separated by # on S's tape
- Mark current location on each tape

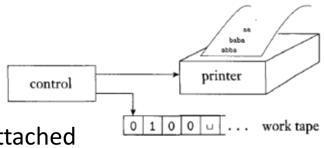
- Read stage: sweep through all *k* tapes to check input
- Write stage: sweep through all *k* tapes to write output **and** update marker (read head) locations
- Head location out of range?
  - Add new position to relevant tape, shift all other characters to right

Equivalence of Deterministic and Nondeterministic TMs

- Try all possible non-deterministic branches – breadth first search
- DTM accepts if NTM accepts
- Can use three tapes: 1 for input, 1 for current branch, 1 to track tree position

29

Enumerators



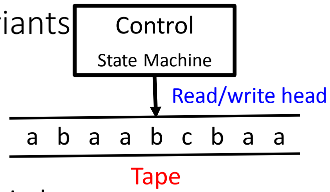
Enumerator E is TM with printer attached

- TM can send strings to be output by printer
- Input tape starts blank
- Language enumerated by E is collection of strings printed
- E may print infinitely

Theorem: A language is Turing-recognizable iff some enumerator enumerates it

30

Common themes in TM variants



- Unlimited access to unlimited memory
- Finite work performed at each step

Note, all programming languages are equivalent

- Can write compiler for C++ in Java

31

An Algorithm

is a collection of simple instructions for carrying out some task

32

Hilbert's Problems

In 1900, David Hilbert proposed 23 mathematical problems

Problem #10
- Devise algorithm to determine if a polynomial has an integral root.
- Example: $6x^3yz^2+3xy^2-x^3-10$ has root x=5, y=3, z=0

General algorithm for Problem 10 does not exist!

33

Church-Turing Thesis

- Intuition of thesis: algorithm == corresponding Turing machine
- Algorithm described by TM also can be describe by λ–calculus (devised by Alonzo Church)

34

Hilbert's 10th problem

Is language D decidable, where D={p | p is polynomial with integral root}

Devise procedure:
- Try all ints, starting at 0: x=0, 1, -1, 2, -2, 3, -3, …
- You may never terminate – so not decidable

Exception: univariate case for root **is** decidable

35

Levels of description

For FA and PDA
- Formal or informal description of machine operation

For TM
- Formal or informal description of machine operation
- **OR** just describe algorithm
  - Assume TM confirms input follows proper tape string format

36