CISC 1100/1400
Structures of Comp. Sci./Discrete Structures
Chapter 2
Sequences

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Outline

- Finding patterns
- Notation
  - Closed form
  - Recursive form
  - Converting between them
- Summations

What number comes next?

- 1, 2, 3, 4, 5,

What number comes next?

- 1, 2, 3, 4, 5, 6

Sequences: Finding patterns

What number comes next?

- 1, 2, 3, 4, 5, 6
- 2, 6, 10, 14, 18,

Sequences: Finding patterns

What number comes next?

- 1, 2, 3, 4, 5, 6
- 2, 6, 10, 14, 18, 22

Sequences: Finding patterns

What number comes next?

- 1, 2, 3, 4, 5, 6
- 2, 6, 10, 14, 18, 22
- 1, 2, 4, 8, 16,

Sequences: Finding patterns

What number comes next?

- 1, 2, 3, 4, 5, 6
- 2, 6, 10, 14, 18, 22
- 1, 2, 4, 8, 16, 32

Sequences: Finding patterns

What number comes next?

- 1, 2, 3, 4, 5, 6
- 2, 6, 10, 14, 18, 22
- 1, 2, 4, 8, 16, 32
- 1, 3, 6, 10, 15,

What number comes next?

- 1, 2, 3, 4, 5, 6
- 2, 6, 10, 14, 18, 22
- 1, 2, 4, 8, 16, 32
- 1, 3, 6, 10, 15, 21

Sequences: Finding patterns

What number comes next?

- 1, 2, 3, 4, 5, 6
- 2, 6, 10, 14, 18, 22
- 1, 2, 4, 8, 16, 32
- 1, 3, 6, 10, 15, 21
- 1, 2, 6, 24, 120,

Sequences: Finding patterns

What number comes next?

- 1, 2, 3, 4, 5, 6
- 2, 6, 10, 14, 18, 22
- 1, 2, 4, 8, 16, 32
- 1, 3, 6, 10, 15, 21
- 1, 2, 6, 24, 120, 720

Sequences: Finding patterns

What number comes next?

- 1, 2, 3, 4, 5, 6
- 2, 6, 10, 14, 18, 22
- 1, 2, 4, 8, 16, 32
- 1, 3, 6, 10, 15, 21
- 1, 2, 6, 24, 120, 720
- 1, 1, 2, 3, 5, 8, 13,

Sequences: Finding patterns

What number comes next?

- 1, 2, 3, 4, 5, 6
- 2, 6, 10, 14, 18, 22
- 1, 2, 4, 8, 16, 32
- 1, 3, 6, 10, 15, 21
- 1, 2, 6, 24, 120, 720
- 1, 1, 2, 3, 5, 8, 13, 21

Discovering the pattern

- Each term might be related to previous terms
- Each term might depend on its position number (1st, 2nd, 3rd, . . .)
- "Well-known" sequences (even numbers, odd numbers)
- Some (or all) of the above

$2, 4, 6, 8, 10, \ldots$

Can we relate a term to previous terms?

- Second term is 2 more than first term.

$2, 4, 6, 8, 10, \ldots$

Can we relate a term to previous terms?

- Second term is 2 more than first term.
- Third term is 2 more than second term.

$2, 4, 6, 8, 10, \ldots$

Can we relate a term to previous terms?

- Second term is 2 more than first term.
- Third term is 2 more than second term.

  $\vdots$

- Any given term is 2 more than previous term.

$2, 4, 6, 8, 10, \ldots$

Can we describe each term by its position in the sequence?

- Term at position 1 is 2.

$2, 4, 6, 8, 10, \ldots$

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- Term at position 1 is 2.
- Term at position 2 is 4.

$2, 4, 6, 8, 10, \ldots$

Can we describe each term by its position in the sequence?

- Term at position 1 is 2.
- Term at position 2 is 4.
- Term at position 3 is 6.

$2, 4, 6, 8, 10, \dots$

Can we describe each term by its position in the sequence?

- Term at position 1 is 2.
- Term at position 2 is 4.
- Term at position 3 is 6.

  $\vdots$

- Term at position n is $2n$.

Mathematical notation

- Write term in a sequence as a lower case letter, followed by a *subscript* denoting position number of the term (e.g., a_1, b_7, z_k).
- For the sequence $2, 4, 6, 8, 10, \ldots$:
  - $a_1 =$

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  - $a_2 =$

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- For the sequence $2, 4, 6, 8, 10, \ldots$:
  - $a_1 = 2$.
  - $a_2 = 4$.
  - a_n is nth term in the sequence.
- What is a_3?

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- What is a_3? 6

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- For the sequence $2, 4, 6, 8, 10, \ldots$:
  - $a_1 = 2$.
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  - a_n is nth term in the sequence.
- What is a_3? 6
- What is a_5?

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- What is a_3? 6
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- What is a_6?

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  - $a_1 = 2$.
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- What is a_3? 6
- What is a_5? 10
- What is a_6? 12

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- What is a_6? 12
- What is a_n if $n = 5$?

Mathematical notation

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- What is a_{n+1} if $n = 5$?

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Recursive formula

- *Recursive formula* for a sequence: each term is described in relation to previous term(s). For example:

$$a_n = 2a_{n-1}$$

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So

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So

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- Problem: Need a starting point (initial condition) such as

$$a_1 = 1$$

Recursive formula

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- Problem: Need a starting point (initial condition) such as

$$a_1 = 1$$

- So let's try

$$\begin{aligned} a_n &= 2a_{n-1} \qquad \text{for } n \geq 2 \\ a_1 &= 1 \end{aligned}$$

- Example:

$$a_3 = 2a_2 = 2\cdot(2\,a_1) = 4a_1 = 4\cdot 1 = 4$$

Fibonacci sequence

- $1, 1, 2, 3, 5, 8, 13, \dots$

Fibonacci sequence

- $1, 1, 2, 3, 5, 8, 13, \ldots$
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$$\begin{aligned} a_n &= a_{n-1} + a_{n-2} \qquad \text{for } n \geq 3 \\ a_2 &= 1 \\ a_1 &= 1 \end{aligned}$$

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- What's a_{10}? Top-down solution:

$$a_{10} = a_9 + a_8$$

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$$a_{10} = a_9 + a_8 = (a_8 + a_7) + (a_7 + a_6)$$

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Too hard!

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- Better way? Work bottom-up via a grid.

n	1	2	3	4	5	6	7	8	9	10
a_n										

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n	1	2	3	4	5	6	7	8	9	10
a_n	1	1	2							

Fibonacci sequence

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a_n	1	1	2	3						

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n	1	2	3	4	5	6	7	8	9	10
a_n	1	1	2	3	5					

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- Better way? Work bottom-up via a grid.

n	1	2	3	4	5	6	7	8	9	10
a_n	1	1	2	3	5	8	13	21		

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n	1	2	3	4	5	6	7	8	9	10
a_n	1	1	2	3	5	8	13	21	34	

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n	1	2	3	4	5	6	7	8	9	10
a_n	1	1	2	3	5	8	13	21	34	55

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- Fibonacci formula

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a_2 &= 1 \\
a_1 &= 1
\end{aligned}
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- Recursive formula corresponds to "recursive function" in a programming language.
- Fibonacci formula

$$
\begin{aligned}
a_n &= a_{n-1} + a_{n-2} \qquad \text{for } n \geq 3 \\
a_2 &= 1 \\
a_1 &= 1
\end{aligned}
$$

- Recursive function

```
def fib(n):
    if n==1 or n==2:
        return 1
    else:
        return fib(n-1) + fib(n-2)
```

Exercise: Find recursive formula

- $2, 4, 6, 8, 10, \ldots$

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$$
\begin{aligned}
a_n &= a_{n-1} + 2 \qquad \text{for } n \geq 2 \\
a_1 &= 2
\end{aligned}
$$

Exercise: Find recursive formula

- $2, 4, 6, 8, 10, \ldots$

$$\begin{aligned} a_n &= a_{n-1} + 2 \qquad \text{for } n \geq 2 \\ a_1 &= 2 \end{aligned}$$

- $1, 3, 6, 10, 15, \ldots$

Exercise: Find recursive formula

- $2, 4, 6, 8, 10, \ldots$

$$\begin{aligned} a_n &= a_{n-1} + 2 \qquad \text{for } n \geq 2 \\ a_1 &= 2 \end{aligned}$$

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$$\begin{aligned} a_n &= a_{n-1} + n \qquad \text{for } n \geq 2 \\ a_1 &= 1 \end{aligned}$$

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$$\begin{aligned} a_n &= a_{n-1} + n \qquad \text{for } n \geq 2 \\ a_1 &= 1 \end{aligned}$$

- $2, 2, 4, 6, 10, 16, \ldots$

Exercise: Find recursive formula

- $2, 4, 6, 8, 10, \ldots$

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$$\begin{aligned} a_n &= a_{n-1} + n \qquad \text{for } n \geq 2 \\ a_1 &= 1 \end{aligned}$$

- $2, 2, 4, 6, 10, 16, \ldots$

$$\begin{aligned} a_n &= a_{n-1} + a_{n-2} \qquad \text{for } n \geq 3 \\ a_2 &= 2 \\ a_1 &= 2 \end{aligned}$$

Finding a closed formula

- Write each term in relation to its position
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We find

$$r_n = r_{n-1} + 3 \qquad \text{for } n \geq 2$$
$$r_1 = 7$$

Can also use algebraic manipulation. Let's try

$$r_n = 3n + 4$$

again.

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  So

$$r_n - r_{n-1} = (3n+4) - (3n+1) = 3,$$

  i.e.,

$$r_n = r_{n-1} + 3$$

Another example

$$s_n = 2^n - 2$$

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- n at the top: We want to stop summation at the nth term of the sequence
- Portion to the right of the $\Sigma_{i=1}^{n}$: Closed form of sequence we want to sum.

Examples of Σ-notation:

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&\quad +(3\cdot 5+7) \\
&= 10+13+16+19+22 = 80
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$$\sum_{j=2}^{6}(j^2-2) = (2^2-2)+(3^2-2)+(4^2-2)+(5^2-2)+(6^2-2) = 2+7+14+23+34 = 80$$

Note: Parentheses are important!

$$\sum_{i=1}^{5} 3i+7 = (3\cdot 1+3\cdot 2+3\cdot 3+3\cdot 4+3\cdot 5)+7 = 52$$

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Suppose you have a statement $P(n)$ about the positive integer n. How would you prove that $P(n)$ is true for *all* $n \in \mathbb{Z}^+$?

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But this doesn't guarantee that $P(n)$ is true for all n; maybe $P(100000001)$ is false!!

Example: Sum of the first n positive integers

Want to show that

$$\sum_{j=1}^{n} j = \tfrac{1}{2}n(n+1) \qquad \forall\, n \in \mathbb{Z}^+,$$

or, if you prefer,

$$1 + 2 + \cdots + n = \tfrac{1}{2}n(n+1) \qquad \forall\, n \in \mathbb{Z}^+.$$

"How on earth did you come up with this formula in the first place?" Later …

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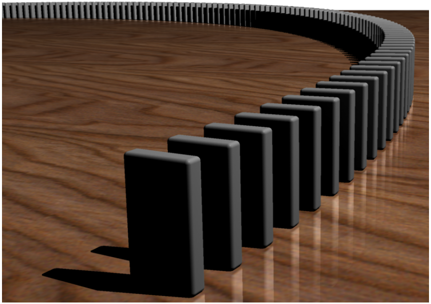

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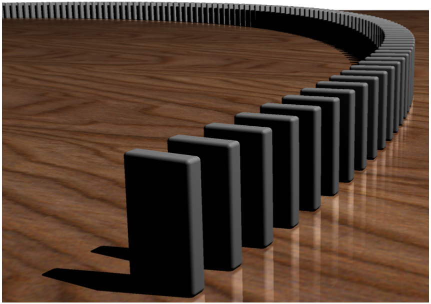

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Let $P(n)$ be a statement about the positive integer $n \in \mathbb{Z}^+$. Suppose we can prove the following:

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... and so on

Example: Sum of the first n positive integers (cont'd)

Theorem

$$\sum_{j=1}^{n} j = \tfrac{1}{2}n(n+1) \qquad \forall\, n \in \mathbb{Z}^+$$

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So formula (1) is true when $n = 1$, i.e., $P(1)$ is true.

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Using this as a starting point, we want to show that $P(k+1)$ is true, i.e., that

$$\sum_{j=1}^{k+1} j = \tfrac{1}{2}(k+1)\Big((k+1)+1\Big) = \tfrac{1}{2}(k+1)(k+2).$$

Induction step (cont'd): But

$$
\begin{aligned}
\sum_{j=1}^{k+1} j &= \left(\sum_{j=1}^{k} j\right) + (k+1) \\
&= \tfrac{1}{2}k(k+1) + (k+1) && \text{by the induction hypothesis} \\
&= \left(\tfrac{1}{2}k + 1\right)(k+1) \\
&= \tfrac{1}{2}(k+2)(k+1),
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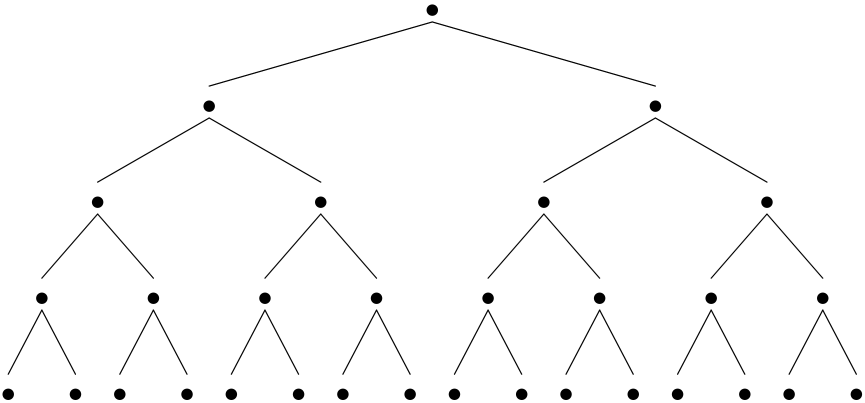
Since we have proved the basis step and the induction step, it follows that $P(n)$ is true for all $n \in \mathbb{Z}^+$. $\square$

Example: Number of leaves in complete binary tree

Here's a complete binary tree with five levels:



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- **Question:** How many nodes branch out from the nth level of a complete binary tree?
- Get an idea by making a table. Let b_n denote the number of nodes branching out from the nth level. Looking at the drawing we saw earlier:

n	1	2	3	4	5
b_n	2	4	8	16	32

- This suggests that $b_n = 2^n$.

Theorem

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Basis step: Let $n = 1$. Looking at the first level of the binary tree, it is immediately clear that $b_1 = 2$. So $P(1)$ is true.

Induction step: Let $k \in \mathbb{Z}^+$, and suppose that $P(k)$ is true; we need to show that $P(k+1)$ is true.

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Hence

$$\begin{aligned} b_{k+1} &= 2b_k \\ &= 2 \cdot 2^k \qquad \text{(by the induction hypothesis)} \\ &= 2^{k+1}, \end{aligned}$$

as required to prove that $P(k+1)$ is true.

Since we have proved the basis step and the induction step, it follows that $P(n)$ is true for all $n \in \mathbb{Z}^+$. $\square$

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