CISC1400: Sets

Fall 2010
Xiaolan Zhang
Dept. of Computer & Information Sciences
Fordham University

1

Outline on sets

- Basics
  - Specify a set by enumerating all elements
  - Notations
  - Cardinality
- Venn Diagram
- Relations on sets: subset, proper subset
- Set builder notation
- Set operations

2

Set: an intuitive definition

- A set is just a collection of objects, these objects are called the members or elements of the set
- One can specify a set by enclosing all its elements with curly braces, separated by commas
- Examples:
  - Set $\{a, b, c, d, e, f\}$ has 6 elements: letter a, letter b, letter c, d, e, and f
  - Set $\{bob, 1, 8, clown, hat\}$ has 5 elements: bob, 1, 8, clown, hat.

3

The elements of a set can be a set itself.

$$\{bob, 1, 8, \{1,2,3\}, a, b, \{clown, bob\}, hat, 4, \{\{\{\{1\}\}\}\}, clown\}$$

4

A set without elements is a special set:

$$\{\}$$

Called the empty set, or null set, often also denoted as ϕ

5

Enumerating set elements

1. You don't list anything more than once

$$\{a, b, a, b, e, f\}$$

2. Order doesn't matter

   The following sets are identical (same)

$$\{1,2,3\} = \{3,1,2\}$$

6

So what if I get tired of writing out all of these Sets?

- Just as with algebra, we give name to a set.
- Typically we use single capital letters to denote a set.
- For example:

$$A = \{a, b, c, d, e, f\}$$

Notations

- Two key symbols that we will see:

$x \in A$ means "x is an element of set A"

$x \notin A$ means "x is not an element of set A"

$$a \in \{a, b, c\}$$

$$1 \notin \{a, b, c\}$$

Cardinality

- The cardinality of a set A is the number elements in the set, denoted as |A|.
- For example:

$$A = \{a, b, c, d, e, f\}$$

$$|A| = 6$$

$$|\{\}| = 0$$

Exercises

1. What is |A|?
   3
2. What is |B|+|C|
   6
3. What is |D|+|E|−|A|?
   3+0−3=0

$A = \{alpha, beta, gamma\}$

$B = \{-5, 0, 5, 10\}$

$C = \{\{a, \{b\}\}, \{c, d\}\}$

$D = \{\{\}, \{\}, 10, 11\}$

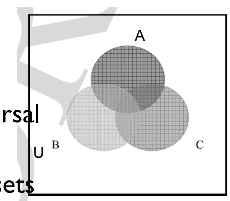
$E = \{\}$

Outline on sets

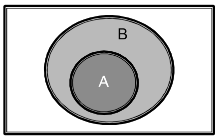
- Basics
  - Specify a set by enumerating all elements
  - Notations
  - Cardinality
- Venn Diagram
- Relations on sets: subset, proper subset
- Set builder notation
- Set operations

Venn Diagram

- Venn Diagram is a diagram for visualizing sets
  - A rectangle represents universal set, U
  - Circles within it represents sets
- Universal set: the set contains all elements that we are interested in
  - Ex: U: the set of all Fordham students
    - A: all freshman students, B: all female students,
    - C: all science major students



Relations between sets



A is totally included in set B, i.e., every element of A is also an element of B, denoted as $A \subseteq B$, read as A is a subset of B

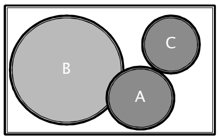
For example:

$\{1,3,5\} \subseteq \{1,2,3,4,5\}$

$\{1,2,3\} \subseteq \{1,2,3\}$ Any set is a subset of itself

$\{\} \subseteq \{1,2,3,4,5\}$ Empty set is subset of any set

Relations between sets



If A is not totally included in set B, i.e., there exists one element of A that is not an element of B, then A is not a subset of B, denoted as $A \not\subseteq B$

For example:

$\{1,3,6\} \not\subseteq \{1,2,3,4,5\}$

$\{1,2,3\} \not\subseteq \{4,5\}$

Proper subset

- We know $\{1,2,3,4,5\} \subseteq \{1,2,3,4,5\}$

  $\{1,3,5\} \subseteq \{1,2,3,4,5\}$
- In second one, there is at least one element in {1,2,3,4,5} that is not an element of {1,3,5}, we say the former is a proper subset of the latter, denoted as $\{1,3,5\} \subset \{1,2,3,4,5\}$
- If A is a subset of B, and A≠B, then A is a proper subset of B, denoted as $A \subset B$
- Analogy to ≤ and < relations between numbers

Exercise

- Find out all subsets of set A={1,2}
- Find out all subsets of set A={a,b,c}
- Find out all proper subsets of set A={a,b,c}

Outline on sets

- Basics
  - Specify a set by enumerating all elements
  - Notations
  - Cardinality
- Venn Diagram
- Relations on sets: subset, proper subset
- Set builder notation
- Set operations

Some well-known sets

- **N** is the natural numbers {0, 1, 2, 3, 4, 5, …}

  $1 \in N \quad -10 \notin N \quad 3.1415 \notin N$
- **Z** is the set of integers {…-2,-1,0,1,2,…}
- **Q** is the set of rational numbers
  - Any number that can be written as a fraction, that is $\frac{p}{q}$, where p and q are integers, and q≠0
  - e.g. $\pi \notin Q \quad \sqrt{2} \notin Q$
- **R** is the set of real numbers
  - all numbers/fractions/decimals that you can imagine, including π, $\sqrt{2}$ etc.

Some Well-known Sets: Variations

- N^+ is the set of positive natural numbers, { 1, 2, 3, 4, 5, …}
- Z^- is the set of negative integers {-1,-2,-3,…}
- $Q^{>1}$ is the set of rational numbers that are greater than 1
- $R^{<10}$ is the set of real numbers that are smaller than 10

19

Set Builder Notation

- We don't always have the ability or want to directly list every element in a set.
- Mathematicians have invented "Set Builder Notation". For example,

$$\{x : x \in N \text{ and } x > 10\}$$

$$\{x \mid x \in N \text{ and } 3x > 10\}$$

read as "a set contains all x's such that x is an element of the set of natural numbers and … "

20

Set Builder Notation

$$\{x : x \in N \text{ and } x > 10\}$$

first half: what we want to include in our set

Second half: constrains on objects specified in first half for it to be an element of the set.

21

Reading set builder notations

$\{x : x \times 2 = 5\}$ $\{2.5\}$

$\{x : x = 2k \text{ and } k \in \{1,2,3\}\}$ $\{2,4,6\}$

$\{x : x \in N \text{ and } \frac{x}{3} \in N\}$

$\{0,3,6,9,12,15,...\}$

$\{x \mid x = 2y \text{ for some } y \in Z^+\}$

$\{2,4,6,8,10,12,...\}$

22

More about set builder

- First half: can be an expression, or specify part of the constraints.
- For example:
  - Let A={1,2,3,5}

$\{x \in A : x \text{ is even}\}$ $= \{2\}$

$\{x + 4 : x \in \{1,2,3\}\}$ $= \{5,6,7\}$

$\{x + y : x \in A \text{ and } y \in \{1,2,3\}\}$

$= \{2,3,4,5,6,7,8\}$

23

Some exercises

- Answer the following questions:

$A = \{x \in Q : x^2 \leq 1\}$

- What is |A| ?
- Are the following true ?

$1.001 \in A$ $0.999 \in A$

- Find all elements of set B defined as follows:

$B = \{2x : x \in N \text{ and } x \leq 4\}$

24

Outline on sets

- Basics
  - Specify a set by enumerating all elements
  - Notations
  - Cardinality
- Venn Diagram
- Relations on sets: subset, proper subset
- Set builder notation
- Set operations

True or False

1. If $x \in A$, and $A \subseteq B$, then $x \in B$
2. If $A \subseteq B$, and $B \subseteq C$, then $A \subseteq C$
3. If $A \subseteq B$, then $|A| \leq |B|$
4. If $|A| \leq |B|$, then $A \subseteq B$
5. {} has no subset.
6. Which of the following are true:

$\{1,2,3\} \subseteq \{\{1,2,3\}\}$

$\{1,2,3\} \in \{\{1,2,3\}\}$

$\{1,2,3\} \in \{\{\{1,2,3\}\}\}$

Review on set

- Let A = {a, {a}, {{a}}}, B = {a}, C = {∅, {a, {a}}}. Which of the following statements are true?

(a) B ⊆ A
(b) B ∈ A
(c) C ⊆ A
(d) ∅ ⊆ C
(e) ∅ ∈ C
(f) {a, {a}} ∈ A
(g) {a, {a}} ⊆ A
(h) B ⊆ C
(i) {{a}} ⊆ A

Set Operations

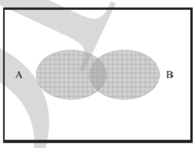
- Just like in arithmetic, there are lots of ways we can perform operation on sets. Most of these operations are different ways of combining two different sets, but some (like Cardinality) only apply to a single set.

Union

$$A \cup B$$

Create a new set by combining all of the elements or two sets, i.e.,

$$A \cup B := \{x \mid x \in \text{A or } x \in B\}$$



The part that has been shaded.

Union Examples

$A = \{1,2,3,4,5\}$ $A \cup B = \{0,1,2,3,4,5,6,8\}$

$B = \{0,2,4,6,8\}$ $B \cup A = \{0,1,2,3,4,5,6,8\}$

$C = \{0,5,10,15\}$ $C \cup D = \{0,5,10,15\}$

$D = \{\}$

$(A \cup C) \cup (D \cup B) = \{0,1,2,3,4,5,6,8,10,15\}$

Intersection

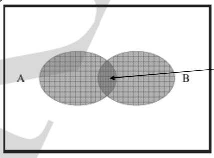
$$A \cap B$$

Create a new set using the elements the two sets have if common

$$A \cap B := \{x \mid x \in \text{A and } x \in B\}$$

"is defined as"

The part that has been shaded twice is $A \cap B$



31

Intersection Examples

$A = \{1,2,3,4,5\}$ $A \cap B = \{2,4\}$

$B = \{0,2,4,6,8\}$ $\{2,4\}$

$C = \{0,5,10,15\}$ $C \cap D = \{\}$

$D = \{\}$

$(A \cap C) \cup (D \cap B) = \{5\}$

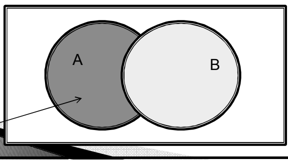
32

Difference

$$A - B$$

Create a new set that includes all elements of set A, removing those elements that are also in set B

$$A - B := \{x \mid x \in \text{A and } x \notin B\}$$



33

Difference Examples

$A = \{1,2,3,4,5\}$ $A - B = \{1,3,5\}$

$B = \{0,2,4,6,8\}$ $\{0,6,8\}$

$C = \{0,5,10,15\}$ $C - D = \{0,5,10,15\}$

$D = \{\}$

$(A - C) - (D - B) = \{1,2,3,4\}$

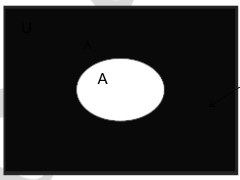
34

Complement

Universal set: the set that includes everything

The difference of U and A is also called the complement of A:

$$A^c := U - A = \{x \mid x \in U \text{ and } x \notin A\}$$



Colored area is U-A

35

Set operations example

- A is the set of all computer science majors.
- B is the set of all physics majors.
- C is the set of all science majors.
- D is the set of all female students.
- Using set operations, describe each of the following in terms of the sets A, B, C and D:
  - Set of all male physics majors.
  - Set of all students who are female or science majors.
  - Set of all students not majoring in science.

36

Power Set

$$P(A)$$

is a set that consists of all subsets of set A.

$$P(A) := \{x : x \subseteq A\}$$

e.g. P({1})=?

List all subsets of {1}: {},{1}

Therefore P({1})={{},{1}}.

37

Power Set Examples

$A = \{1,2,3\}$

$B = \{a,b,c,d\}$

$$P(A) = \{\{1,2,3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1\}, \{2\}, \{3\}, \{\}\}$$

$$P(B) = \{\{a,b,c,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}, \{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}, \{a\}, \{b\}, \{c\}, \{d\}, \{\}\}$$

38

Exercises on power set

1. $A = \{\}$

$P(A) =$

2. $C = \{a, 1, \{\}\}$

$P(C) =$

39

Cardinality of Power Set

- If |A|=1, |P(A)|= ?
  - Try P({a})=
- If |A|=2, |P(A)|=
  - Try P({a,b})
- If set A have a certain number of subsets, after we add one more element into A, how many subsets A has now ?
  - Every originally identified subsets are still valid
  - Add the new element into each of them, and we get a new subset.
  - The number of subsets doubles !

40

Cardinality of Power Set

- Let a_n be the number of subsets that a set of cardinality n has, then

$$a_1 = 2$$
$$a_n = 2a_{n-1}$$

- We can find the closed form:

$$a_n = 2^n$$

- A set of cardinality n has 2^n subsets
- If |A|=n, |P(A)|=2^n

41

Cartesian Product (Cross Product)

$$A \times B$$

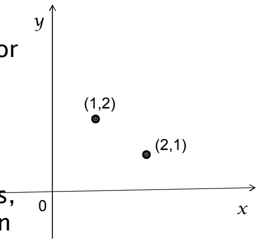
Create a new set consisting of all possible ordered pairs where the first element is from A, and second element is from B.

$$A \times B := \{(x, y) : x \in A \text{ and } y \in B\}$$

42

Ordered Pair (x, y)

- Its just like what you learned about when you learned about graphing or points
  $(1,2) \neq (2,1)$
- It's different from set !
  - {1,2}={2,1}
- x, and y can be numbers, names, anything you can imagine



43

Example of Cartesian Product

- I have two T-shirts: white, black
  - A={white shirt, black shirt}
- I have three jeans: black, blue, green
  - B={black jean, blue jean, green jean}
- All outfits I can make out of these ?
  - The set of all ordered-pairs, in the form (T-shirt, jean)…

44

Cartesian Product Examples

$A = \{1,2,3\}$

$B = \{a,b,c\}$

$C = \{-1,5\}$

$A \times B = \{(1,a),(1,b),(1,c),(2,a),(2,b),(2,c),(3,a),(3,b),(3,c)\}$

$B \times A =$ $\{(a,1),(a,2),(a,3),(b,1),(b,2),(b,3),(c,1),(c,2),(c,3)\}$

$C \times A = \{(-1,1),(-1,2),(-1,3),(5,1),(5,2),(5,3)\}$

$B \times C = \{(a,-1),(a,5),(b,-1),(b,5),(c,-1),(c,5)\}$

45

Cardinality of Cartesian Product

- If A has m elements, B has n elements, how many elements does AxB have ?
  - For every element of A, we pair it with each of the n elements in B,. to get n ordered pairs in AxB
  - So we can form n*m ordered pairs this way
  - So |AxB| = m*n = |A|*|B|
- This is where the name Cartesian product comes from.

46

Exercises on Power Set/Cartesian Product

- {} x {1,2}=
- P({a,b}) x {c,d} =
- Can you find a set A, such that P(A)={{}, {2},{1}} ?
- Is it true that for any set A, $\{\} \in P(A)?$
- Is it true that for any set A, $\{\} \subseteq P(A)?$

47

Summary

- Basics
  - Specify a set by enumerating all elements
  - Notations
  - Cardinality
- Venn Diagram
- Relations on sets: subset, proper subset
- Set builder notation
- Set operations

48