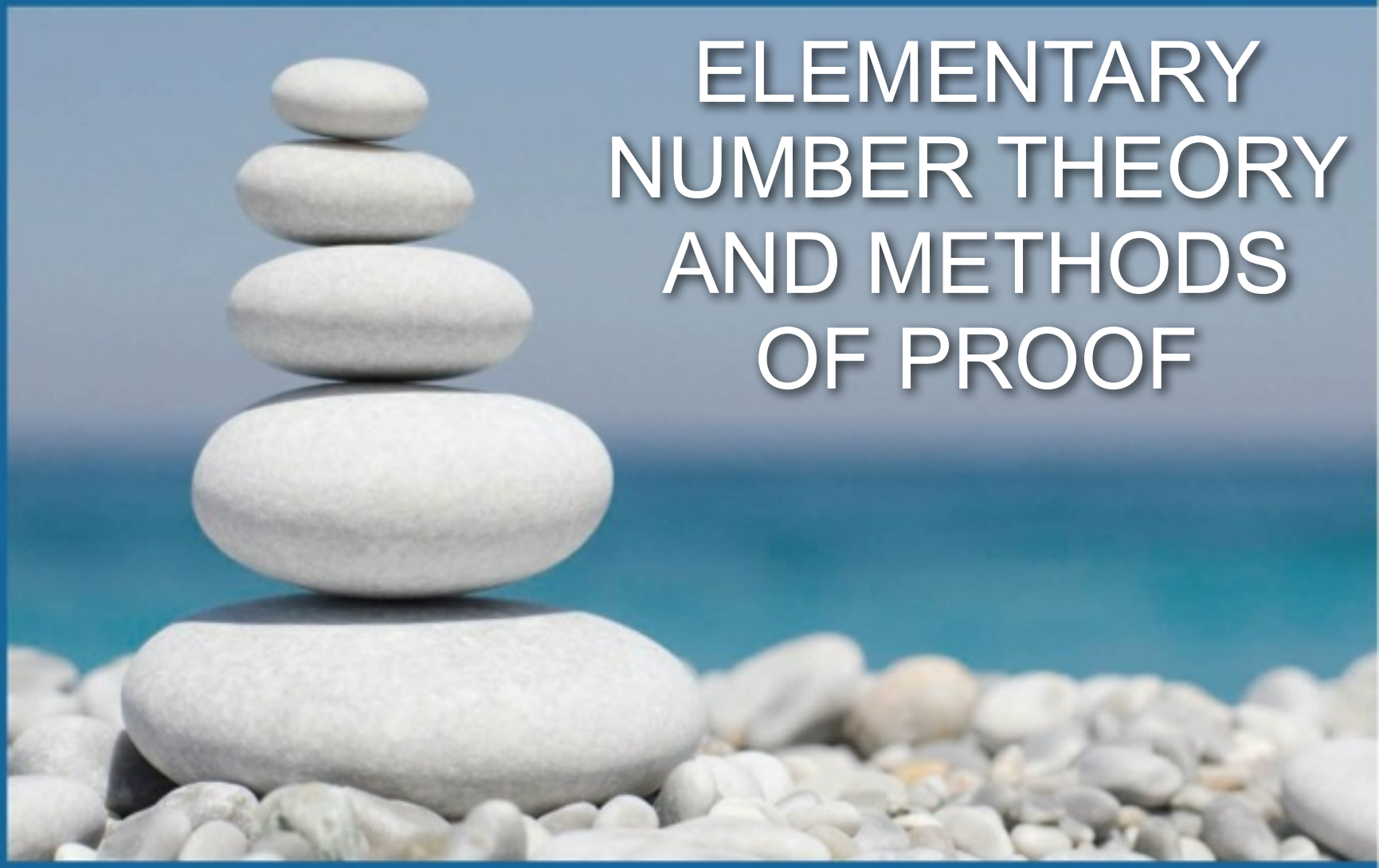
CHAPTER 4



ELEMENTARY NUMBER THEORY AND METHODS OF PROOF

SECTION 4.6

Indirect Argument: Contradiction and Contraposition

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In a direct proof you start with the hypothesis of a statement and make one deduction after another until you reach the conclusion.

Indirect proofs are more roundabout. One kind of indirect proof, *argument by contradiction*, is based on the fact that either a statement is true or it is false but not both.

So if you can show that the assumption that a given statement is not true leads logically to a contradiction, impossibility, or absurdity, then that assumption must be false: and, hence, the given statement must be true.

Indirect Argument: Contradiction and Contraposition

This method of proof is also known as *reductio ad impossible or reductio ad absurdum* because it relies on reducing a given assumption to an impossibility or absurdity.

Indirect Argument: Contradiction and Contraposition

The point of departure for a proof by contradiction is the supposition that the statement to be proved is false. The goal is to reason to a contradiction. Thus proof by contradiction has the following outline:

Method of Proof by Contradiction

1. Suppose the statement to be proved is false. That is, suppose that the negation of the statement is true.
2. Show that this supposition leads logically to a contradiction.
3. Conclude that the statement to be proved is true.

Example 1 – *There Is No Greatest Integer*

Use proof by contradiction to show that there is no greatest integer.

Solution:

Most small children believe there is a greatest integer—they often call it a “zillion.”

But with age and experience, they change their belief. At some point they realize that if there were a greatest integer, they could add 1 to it to obtain an integer that was greater still.

Since that is a contradiction, no greatest integer can exist. This line of reasoning is the heart of the formal proof.

Example 1 – *Solution*

cont'd

For the proof, the "certain property" is the property of being the greatest integer. To prove that there is no object with this property, begin by supposing the negation: that there is an object with the property.

Starting Point: Suppose not. Suppose there is a greatest integer; call it N. This means that $N \geq n$ for all integers n.

To Show: This supposition leads logically to a contradiction.

Example 1 – *Solution*

cont'd

Theorem 4.6.1

There is no greatest integer.

Proof:

[*We take the negation of the theorem and suppose it to be true.*] Suppose not. That is, suppose there is a greatest integer *N*. [*We must deduce a contradiction.*]

Example 1 – *Solution*

cont'd

Then $N \geq n$ for every integer n. Let $M = N + 1$. Now M is an integer since it is a sum of integers. Also $M > N$ since $M = N + 1$. Thus M is an integer that is greater than N.

So N is the greatest integer and N is not the greatest integer, which is a contradiction. *[This contradiction shows that the supposition is false and, hence, that the theorem is true.]*

Indirect Argument: Contradiction and Contraposition

The fact that no integer can be both even and odd follows from the uniqueness part of the quotient-remainder theorem.

Theorem 4.6.2

There is no integer that is both even and odd.

Argument by Contraposition

Argument by Contraposition

A second form of indirect argument, *argument by contraposition*, is based on the logical equivalence between a statement and its contrapositive.

To prove a statement by contraposition, you take the contrapositive of the statement, prove the contrapositive by a direct proof, and conclude that the original statement is true.

The underlying reasoning is that since a conditional statement is logically equivalent to its contrapositive, if the contrapositive is true then the statement must also be true.

Argument by Contraposition

Method of Proof by Contraposition

1. Express the statement to be proved in the form

   $$\forall x \text{ in } D, \text{ if } P(x) \text{ then } Q(x).$$

   (This step may be done mentally.)

2. Rewrite this statement in the contrapositive form

   $$\forall x \text{ in } D, \text{ if } Q(x) \text{ is false then } P(x) \text{ is false.}$$

   (This step may also be done mentally.)

3. Prove the contrapositive by a direct proof.

   a. Suppose x is a (particular but arbitrarily chosen) element of D such that $Q(x)$ is false.

   b. Show that $P(x)$ is false.

Example 4 – *If the Square of an Integer Is Even, Then the Integer Is Even*

Prove that for all integers n, if n^2 is even then n is even.

Solution:
First form the contrapositive of the statement to be proved.

Contrapositive: For all integers n, if n is not even then n^2 is not even.

By the quotient-remainder theorem with $d = 2$, any integer is even or odd, so any integer that is not even is odd. Also by Theorem 4.6.2, no integer can be both even and odd. So if an integer is odd, then it is not even.

Example 4 – *Solution*

cont'd

Thus the contrapositive can be restated as follows:

Contrapositive: For all integers n, if n is odd then n^2 is odd.

A straightforward computation is the heart of a direct proof for this statement, which is as follows.

Proposition 4.6.4

For all integers n, if n^2 is even then n is even.

Example 4 – *Solution*

cont'd

Proof (by contraposition):
Suppose n is any odd integer. *[We must show that n^2 is odd.]* By definition of odd, $n = 2k + 1$ for some integer k. By substitution and algebra,

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1.$$

But $2k^2 + 2k$ is an integer because products and sums of integers are integers.

So $n^2 = 2 \bullet$ (an integer) + 1, and thus, by definition of odd, n^2 is odd *[as was to be shown].*

Example 4 – *Solution*

cont'd

We used the word *proposition* here rather than *theorem* because although the word *theorem* can refer to any statement that has been proved, mathematicians often restrict it to especially important statements that have many and varied consequences.

Then they use the word **proposition** to refer to a statement that is somewhat less consequential but nonetheless worth writing down.

Relation between Proof by Contradiction and Proof by Contraposition

Relation between Proof by Contradiction and Proof by Contraposition

Observe that any proof by contraposition can be recast in the language of proof by contradiction. In a proof by contraposition, the statement

$$\forall x \text{ in } D, \text{ if } P(x) \text{ then } Q(x)$$

is proved by giving a direct proof of the equivalent statement

$$\forall x \text{ in } D, \text{ if } \sim Q(x) \text{ then } \sim P(x).$$

Relation between Proof by Contradiction and Proof by Contraposition

To do this, you suppose you are given an arbitrary element x of D such that $\sim Q(x)$. You then show that $\sim P(x)$. This is illustrated in Figure 4.6.1.

Proof by Contraposition

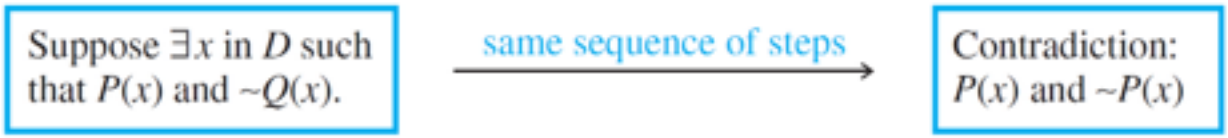
Figure 4.6.1

Exactly the same sequence of steps can be used as the heart of a proof by contradiction for the given statement. The only thing that changes is the context in which the steps are written down.

Relation between Proof by Contradiction and Proof by Contraposition

To rewrite the proof as a proof by contradiction, you suppose there is an x in D such that $P(x)$ and $\sim Q(x)$.

You then follow the steps of the proof by contraposition to deduce the statement $\sim P(x)$. But $\sim P(x)$ is a contradiction to the supposition that $P(x)$ and $\sim Q(x)$. (Because to contradict a conjunction of two statements, it is only necessary to contradict one of them.) This process is illustrated in Figure 4.6.2.



Proof by Contradiction

Figure 4.6.2

Relation between Proof by Contradiction and Proof by Contraposition

As an example, here is a proof by contradiction of Proposition 4.6.4, namely that for any integer n, if n^2 is even then n is even.

Proposition 4.6.4

For all integers n, if n^2 is even then n is even.

Proof (by contradiction):
[We take the negation of the theorem and suppose it to be true.] Suppose not. That is, suppose there is an integer n such that n^2 is even and n is not even. *[We must deduce a contradiction.]*

Relation between Proof by Contradiction and Proof by Contraposition

By the quotient-remainder theorem with $d = 2$, any integer is even or odd. Hence, since n is not even it is odd, and thus, by definition of odd, $n = 2k + 1$ for some integer k. By substitution and algebra:

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1.$$

But $2k^2 + 2k$ is an integer because products and sums of integers are integers.

So $n^2 = 2 \bullet$ (an integer) + 1, and thus, by definition of odd, n^2 is odd. Therefore, n^2 is both even and odd.

Relation between Proof by Contradiction and Proof by Contraposition

This contradicts Theorem 4.6.2, which states that no integer can be both even and odd.

[This contradiction shows that the supposition is false and, hence, that the proposition is true.]

Relation between Proof by Contradiction and Proof by Contraposition

Note that when you use proof by contraposition, you know exactly what conclusion you need to show, namely the negation of the hypothesis; whereas in proof by contradiction, it may be difficult to know what contradiction to head for.

On the other hand, when you use proof by contradiction, once you have deduced any contradiction whatsoever, you are done.

The main advantage of contraposition over contradiction is that you avoid having to take (possibly incorrectly) the negation of a complicated statement.

Relation between Proof by Contradiction and Proof by Contraposition

The disadvantage of contraposition as compared with contradiction is that you can use contraposition only for a specific class of statements—those that are universal and conditional.

The previous discussion shows that any statement that can be proved by contraposition can be proved by contradiction. But the converse is not true.

Statements such as "$\sqrt{2}$ is irrational" can be proved by contradiction but not by contraposition.

Proof as a Problem-Solving Tool

Proof as a Problem-Solving Tool

Direct proof, disproof by counterexample, proof by contradiction, and proof by contraposition are all tools that may be used to help determine whether statements are true or false. Given a statement of the form

> For all elements in a domain, if (hypothesis) then (conclusion),

imagine elements in the domain that satisfy the hypothesis. Ask yourself: Must they satisfy the conclusion?

Proof as a Problem-Solving Tool

If you can see that the answer is “yes” in all cases, then the statement is true and your insight will form the basis for a direct proof.

If after some thought it is not clear that the answer is “yes,” ask yourself whether there are elements of the domain that satisfy the hypothesis and *not* the conclusion.

If you are successful in finding some, then the statement is false and you have a counterexample. On the other hand, if you are not successful in finding such elements, perhaps none exist.

Proof as a Problem-Solving Tool

Perhaps you can show that assuming the existence of elements in the domain that satisfy the hypothesis and not the conclusion leads logically to a contradiction.

If so, then the given statement is true and you have the basis for a proof by contradiction. Alternatively, you could imagine elements of the domain for which the conclusion is false and ask whether such elements also fail to satisfy the hypothesis.

If the answer in all cases is “yes,” then you have a basis for a proof by contraposition.