Proposition logic and argument

SECTION 2.1

Logical Form and Logical Equivalence

Statements

Statements

Most of the definitions of formal logic have been developed so that they agree with the natural or intuitive logic used by people who have been educated to think clearly and use language carefully.

The differences that exist between formal and intuitive logic are necessary to avoid ambiguity and obtain consistency.

In any mathematical theory, new terms are defined by using those that have been previously defined. However, this process has to start somewhere. A few initial terms necessarily remain undefined.

Statements

In logic, the words *sentence*, *true*, and *false* are the initial undefined terms.

• Definition

A **statement** (or **proposition**) is a sentence that is true or false but not both.

Compound Statements

Compound Statements

We now introduce three symbols that are used to build more complicated logical expressions out of simpler ones.

The symbol ~ denotes *not*, ∧ denotes *and*, and ∨ denotes *or*.

Given a statement *p*, the sentence "~*p*" is read "not *p*" or "It is not the case that *p*" and is called the **negation of *p*.** In some computer languages the symbol ⌐ is used in place of ~.

Compound Statements

Given another statement q, the sentence "$p \wedge q$" is read "p and q" and is called the **conjunction of *p* and *q***.

The sentence "$p \vee q$" is read "p or q" and is called the **disjunction of *p* and *q***.

In expressions that include the symbol ~ as well as $\wedge$ or $\vee$, the **order of operations** specifies that ~ is performed first.

For instance, $\sim p \wedge q = (\sim p) \wedge q$.

Compound Statements

In logical expressions, as in ordinary algebraic expressions, the order of operations can be overridden through the use of parentheses.

Thus $\sim(p \wedge q)$ represents the negation of the conjunction of p and q.

In this, as in most treatments of logic, the symbols $\wedge$ and $\vee$ are considered coequal in order of operation, and an expression such as $p \wedge q \vee r$ is considered ambiguous.

This expression must be written as either $(p \wedge q) \vee r$ or $p \wedge (q \vee r)$ to have meaning.

Example 2 – *Translating from English to Symbols: But and Neither-Nor*

Write each of the following sentences symbolically, letting h = "It is hot" and s = "It is sunny."

a. It is not hot but it is sunny.

b. It is neither hot nor sunny.

Solution:

a. The given sentence is equivalent to "It is not hot and it is sunny," which can be written symbolically as $\sim h \wedge s$.

b. To say it is neither hot nor sunny means that it is not hot and it is not sunny. Therefore, the given sentence can be written symbolically as $\sim h \wedge \sim s$.

Truth Values

Truth Values

In Example 2 we built compound sentences out of component statements and the terms *not*, *and*, and *or*.

If such sentences are to be statements, however, they must have well-defined **truth values**—they must be either true or false.

We now define such compound sentences as statements by specifying their truth values in terms of the statements that compose them.

Truth Values

The negation of a statement is a statement that exactly expresses what it would mean for the statement to be false.

• Definition

If p is a statement variable, the **negation** of p is "not p" or "It is not the case that p" and is denoted $\sim p$. It has opposite truth value from p: if p is true, $\sim p$ is false; if p is false, $\sim p$ is true.

The truth values for negation are summarized in a *truth table*.

p	$\sim p$
T	F
F	T

Truth Table for ~*p*

Truth Values

Definition

If p and q are statement variables, the **conjunction** of p and q is "p and q," denoted $p \wedge q$. It is true when, and only when, both p and q are true. If either p or q is false, or if both are false, $p \wedge q$ is false.

The truth values for conjunction can also be summarized in a truth table.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Truth Table for $p \wedge q$

Truth Values

• Definition

If p and q are statement variables, the **disjunction** of p and q is "p or q," denoted $p \vee q$. It is true when either p is true, or q is true, or both p and q are true; it is false only when both p and q are false.

Here is the truth table for disjunction:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Truth Table for $p \vee q$

Evaluating the Truth of More General Compound Statements

Evaluating the Truth of More General Compound Statements

Now that truth values have been assigned to $\sim p$, $p \wedge q$, and $p \vee q$, consider the question of assigning truth values to more complicated expressions such as $\sim p \vee q$, $(p \vee q) \wedge \sim(p \wedge q)$, and $(p \wedge q) \vee r$. Such expressions are called *statement forms* (or *propositional forms*).

• Definition

A **statement form** (or **propositional form**) is an expression made up of statement variables (such as p, q, and r) and logical connectives (such as $\sim$, $\wedge$, and $\vee$) that becomes a statement when actual statements are substituted for the component statement variables. The **truth table** for a given statement form displays the truth values that correspond to all possible combinations of truth values for its component statement variables.

Example 4 – *Truth Table for Exclusive Or*

Construct the truth table for the statement form
$(p \vee q) \wedge \sim(p \wedge q)$.

Note that when *or* is used in its exclusive sense, the statement "*p* or *q*" means "*p* or *q* but not both" or "*p* or *q* and not both *p* and *q*," which translates into symbols as
$(p \vee q) \wedge \sim(p \wedge q)$.

This is sometimes abbreviated

$$p \oplus q \text{ or } p \text{ XOR } q.$$

Example 4 –

p	q	$p \vee q$	$p \wedge q$	$\sim(p \wedge q)$	$(p \vee q) \wedge \sim(p \wedge q)$
T	T	T	T	F	F
T	F	T	F	T	T
F	T	T	F	T	T
F	F	F	F	T	F

1. Set up columns labeled p, q, $p \vee q$, $p \wedge q$, $\sim(p \wedge q)$, and $(p \vee q) \wedge \sim(p \wedge q)$.
2. Fill in p and q columns with all the logically possible combinations of T's and F's.
3. Use truth tables for $\vee$ and $\wedge$ to fill in the $p \vee q$ and $p \wedge q$ columns with appropriate truth values.
4. Fill in the $\sim(p \wedge q)$ column by taking the opposites of the truth values for $p \wedge q$, *e.g.,* the entry for $\sim(p \wedge q)$ in the first row is F because in the first row the truth value of $p \wedge q$ is T.
5. Fill in the $(p \vee q) \wedge \sim(p \wedge q)$ column by considering the truth table for an *and* statement together with the computed truth values for $p \vee q$ and $\sim(p \wedge q)$.

Logical Equivalence

Logical Equivalence

The statements

6 is greater than 2 and 2 is less than 6

are two different ways of saying the same thing. Why? Because of the definition of the phrases *greater than* and *less than*. By contrast, although the statements

(1) Dogs bark and cats meow

and

(2) Cats meow and dogs bark

are also two different ways of saying the same thing, the reason has nothing to do with the definition of the words. It has to do with the logical form of the statements.

Logical Equivalence

Any two statements whose logical forms are related in the same way as (1) and (2) would either both be true or both be false.

You can see this by examining the following truth table, where the statement variables *p* and *q* are substituted for the component statements “Dogs bark” and “Cats meow,” respectively.

Logical Equivalence

The table shows that for each combination of truth values for p and q, $p \wedge q$ is true when, and only when, $q \wedge p$ is true.

In such a case, the statement forms are called *logically equivalent*, and we say that (1) and (2) are *logically equivalent statements*.

p	q	$p \wedge q$	$q \wedge p$
T	T	T	T
T	F	F	F
F	T	F	F
F	F	F	F

$p \wedge q$ and $q \wedge p$ always have the same truth values, so they are logically equivalent

Logical Equivalence

- **Definition**

Two *statement forms* are called **logically equivalent** if, and only if, they have identical truth values for each possible substitution of statements for their statement variables. The logical equivalence of statement forms P and Q is denoted by writing $P \equiv Q$.

Two *statements* are called **logically equivalent** if, and only if, they have logically equivalent forms when identical component statement variables are used to replace identical component statements.

Logical Equivalence

Testing Whether Two Statement Forms *P* and *Q* Are Logically Equivalent

1. Construct a truth table with one column for the truth values of *P* and another column for the truth values of *Q*.

2. Check each combination of truth values of the statement variables to see whether the truth value of *P* is the same as the truth value of *Q*.

a. If in each row the truth value of *P* is the same as the truth value of *Q*, then *P* and *Q* are logically equivalent.

b. If in some row *P* has a different truth value from *Q*, then *P* and *Q* are not logically equivalent.

Example 6 – *Double Negative Property:* $\sim(\sim p) \equiv p$

Construct a truth table to show that the negation of the negation of a statement is logically equivalent to the statement, annotating the table with a sentence of explanation.

Solution:

p	$\sim p$	$\sim(\sim p)$
T	F	T
F	T	F

p and $\sim(\sim p)$ always have the same truth values, so they are logically equivalent

Logical Equivalence

There are two ways to show that statement forms P and Q are *not* logically equivalent. As indicated previously, one is to use a truth table to find rows for which their truth values differ.

The other way is to find concrete statements for each of the two forms, one of which is true and the other of which is false.

The next example illustrates both of these ways.

Example 7 – *Showing Nonequivalence*

Show that the statement forms $\sim(p \wedge q)$ and $\sim p \wedge \sim q$ are not logically equivalent.

Solution:

a. This method uses a truth table annotated with a sentence of explanation.

p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim(p \wedge q)$		$\sim p \wedge \sim q$
T	T	F	F	T	F		F
T	F	F	T	F	T	$\neq$	F
F	T	T	F	F	T	$\neq$	F
F	F	T	T	F	T		T

$\sim(p \wedge q)$ and $\sim p \wedge \sim q$ have different truth values in rows 2 and 3, so they are not logically equivalent

Example 7 – *Solution*

cont'd

b. This method uses an example to show that ~(p ∧ q) and ~p ∧ ~q are not logically equivalent.

Let p be the statement "0 < 1" and let q be the statement "1 < 0."

Then $\sim(p \wedge q)$ is "It is not the case that both $0 < 1$ and $1 < 0$,"

which is true.

On the other hand, $\sim p \wedge \sim q$ is "$0 \not< 1$ and $1 \not< 0$,"

which is false.

Example 7 – *Solution*

cont'd

This example shows that there are concrete statements you can substitute for *p* and *q* to make one of the statement forms true and the other false.

Therefore, the statement forms are not logically equivalent.

Logical Equivalence

The two logical equivalences are known as **De Morgan's laws** of logic in honor of Augustus De Morgan, who was the first to state them in formal mathematical terms.

De Morgan's Laws

The negation of an *and* statement is logically equivalent to the *or* statement in which each component is negated.

The negation of an *or* statement is logically equivalent to the *and* statement in which each component is negated.

Logical Equivalence

Symbolically we can represent the two logic equivalences as:

$$\boxed{\sim(p \wedge q) \equiv \sim p \vee \sim q}$$

and

$$\boxed{\sim(p \vee q) \equiv \sim p \wedge \sim q.}$$

Example 9 – *Applying De Morgan's Laws*

Write negations for each of the following statements:

a. John is 6 feet tall and he weighs at least 200 pounds.

b. The bus was late or Tom's watch was slow.

Solution:

a. John is not 6 feet tall or he weighs less than 200 pounds.

b. The bus was not late and Tom's watch was not slow.

Since the statement "neither p nor q" means the same as "$\sim p$ and $\sim q$," an alternative answer for (b) is "Neither was the bus late nor was Tom's watch slow."

Tautologies and Contradictions

Tautologies and Contradictions

• Definition

A **tautology** is a statement form that is always true regardless of the truth values of the individual statements substituted for its statement variables. A statement whose form is a tautology is a **tautological statement.**

A **contradication** is a statement form that is always false regardless of the truth values of the individual statements substituted for its statement variables. A statement whose form is a contradication is a **contradictory statement.**

the truth of a tautological statement and the falsity of a contradictory statement are due to the logical structure of the statements themselves and are independent of the meanings of the statements.

Example 13 – *Logical Equivalence Involving Tautologies and Contradictions*

If **t** is a tautology and **c** is a contradiction, show that
$p \wedge \mathbf{t} \equiv p$ and $p \wedge \mathbf{c} \equiv \mathbf{c}$.

Solution:

p	$\mathbf{t}$	$p \wedge \mathbf{t}$	p	$\mathbf{c}$	$p \wedge \mathbf{c}$
T	T	T	T	F	F
F	T	F	F	F	F

same truth values, so $p \wedge \mathbf{t} \equiv p$

same truth values, so $p \wedge \mathbf{c} \equiv \mathbf{c}$

Summary of Logical Equivalences

Summary of Logical Equivalences

Knowledge of logically equivalent statements is very useful for constructing arguments.

It often happens that it is difficult to see how a conclusion follows from one form of a statement, whereas it is easy to see how it follows from a logically equivalent form of the statement.

Summary of Logical Equivalences

Theorem 2.1.1 Logical Equivalences

Given any statement variables p, q, and r, a tautology $\mathbf{t}$ and a contradiction $\mathbf{c}$, the following logical equivalences hold.

1. *Commutative laws:*	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
2. *Associative laws:*	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
3. *Distributive laws:*	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
4. *Identity laws:*	$p \wedge \mathbf{t} \equiv p$	$p \vee \mathbf{c} \equiv p$
5. *Negation laws:*	$p \vee {\sim}p \equiv \mathbf{t}$	$p \wedge {\sim}p \equiv \mathbf{c}$
6. *Double negative law:*	${\sim}({\sim}p) \equiv p$	
7. *Idempotent laws:*	$p \wedge p \equiv p$	$p \vee p \equiv p$
8. *Universal bound laws:*	$p \vee \mathbf{t} \equiv \mathbf{t}$	$p \wedge \mathbf{c} \equiv \mathbf{c}$
9. *De Morgan's laws:*	${\sim}(p \wedge q) \equiv {\sim}p \vee {\sim}q$	${\sim}(p \vee q) \equiv {\sim}p \wedge {\sim}q$
10. *Absorption laws:*	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
11. *Negations of* $\mathbf{t}$ *and* $\mathbf{c}$:	${\sim}\mathbf{t} \equiv \mathbf{c}$	${\sim}\mathbf{c} \equiv \mathbf{t}$

Example 14 – *Simplifying Statement Forms*

Use Theorem 2.1.1 to verify the logical equivalence

$$\sim(\sim p \wedge q) \wedge (p \vee q) \equiv p.$$

Solution:

Use the laws of Theorem 2.1.1 to replace sections of the statement form on the left by logically equivalent expressions.

Each time you do this, you obtain a logically equivalent statement form.

Example 14 – *Solution*

cont'd

Continue making replacements until you obtain the statement form on the right.

$$\begin{aligned}
\sim(\sim p \wedge q) \wedge (p \vee q) &\equiv (\sim(\sim p) \vee \sim q) \wedge (p \vee q) && \text{by De Morgan's laws} \\
&\equiv (p \vee \sim q) \wedge (p \vee q) && \text{by the double negative law} \\
&\equiv p \vee (\sim q \wedge q) && \text{by the distributive law} \\
&\equiv p \vee (q \wedge \sim q) && \text{by the commutative law for } \wedge \\
&\equiv p \vee \mathbf{c} && \text{by the negation law} \\
&\equiv p && \text{by the identity law.}
\end{aligned}$$

SECTION 2.2

Conditional Statements

Conditional Statements

Let *p* and *q* be statements. A sentence of the form "If *p* then *q*" is denoted symbolically by "$p \to q$"; *p* is called the *hypothesis* and *q* is called the *conclusion, e.g.,*

$$\text{If } \underbrace{\text{4,686 is divisible by 6}}_{\text{hypothesis}}\text{, then } \underbrace{\text{4,686 is divisible by 3}}_{\text{conclusion}}$$

Such a sentence is called *conditional* because the truth of statement *q* is conditioned on the truth of statement *p*.

Conditional Statements

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Truth Table for $p \leftrightarrow q$

As is the case with other connectives, formal definition of truth values for →(if-then) is based on its everyday, intuitive meaning.

Manager: "If you show up for work Monday morning, then you will get the job"
When will you be able to say that the manager lies?

A conditional statement that is true by virtue of the fact that its hypothesis is false is often called **vacuously true** or **true by default**.

Thus the statement is vacuously true if you do not show up for work Monday morning.

In general, when the "if" part of an if-then statement is false, the statement as a whole is said to be true, regardless of whether the conclusion is true or false.

Conditional Statements

- **Definition**

If p and q are statement variables, the **conditional** of q by p is "If p then q" or "p implies q" and is denoted $p \to q$. It is false when p is true and q is false; otherwise it is true. We call p the **hypothesis** (or **antecedent**) of the conditional and q the **conclusion** (or **consequent**).

Example 1 – *A Conditional Statement with a False Hypothesis*

Consider the statement:

If 0 = 1 then 1 = 2.

As strange as it may seem, since the hypothesis of this statement is false, the statement as a whole is true.

Conditional Statements

In expressions that include $\rightarrow$ as well as other logical operators such as $\wedge$, $\vee$, and $\sim$, the **order of operations** is that $\rightarrow$ is performed last.

Thus, according to the specification of order of operations, $\sim$ is performed first, then $\wedge$ and $\vee$, and finally $\rightarrow$.

Example 2 – *Truth Table for* $p \vee {\sim}q \rightarrow {\sim}p$

Construct a truth table for the statement form $p \vee {\sim}q \rightarrow {\sim}p$.

Solution:

By the order of operations given above, the following two expressions are equivalent: $p \vee {\sim}q \rightarrow {\sim}p$ and $(p \vee ({\sim}q)) \rightarrow ({\sim}p)$, and this order governs the construction of the truth table.

First fill in the four possible combinations of truth values for p and q, and then enter the truth values for ${\sim}p$ and ${\sim}q$ using the definition of negation.

Example 2 – *Solution*

cont'd

Next fill in the $p \vee {\sim}q$ column using the definition of $\vee$. Finally, fill in the $p \vee {\sim}q \rightarrow {\sim}p$ column using the definition of $\rightarrow$.

The only rows in which the hypothesis $p \vee {\sim}q$ is true and the conclusion ${\sim}p$ is false are the first and second rows.

So you put F's in those two rows and T's in the other two rows.

p	q	$\underbrace{{\sim}p}_{\text{conclusion}}$	${\sim}q$	$\underbrace{p \vee {\sim}q}_{\text{hypothesis}}$	$p \vee {\sim}q \rightarrow {\sim}p$
T	T	F	F	T	F
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

Logical Equivalences Involving $\rightarrow$

Example 3 – *Division into Cases: Showing that* $p \vee q \rightarrow r \equiv (p \rightarrow r) \wedge (q \rightarrow r)$

Use truth tables to show the logical equivalence of the statement forms $p \vee q \rightarrow r$ and $(p \rightarrow r) \wedge (q \rightarrow r)$. Annotate the table with a sentence of explanation.

Solution:

First fill in the eight possible combinations of truth values for p, q, and r.

Then fill in the columns for $p \vee q$, $p \rightarrow r$, and $q \rightarrow r$ using the definitions of *or* and *if-then*.

Example 3 – *Solution*

cont'd

For instance, the $p \rightarrow r$ column has F's in the second and fourth rows because these are the rows in which p is true and q is false.

Next fill in the $p \vee q \rightarrow r$ column using the definition of *if-then*. The rows in which the hypothesis $p \vee q$ is true and the conclusion r is false are the second, fourth, and sixth.

So F's go in these rows and T's in all the others.

Example 3 – *Solution*

cont'd

The complete table shows that $p \vee q \to r$ and $(p \to r) \wedge (q \to r)$ have the same truth values for each combination of truth values of *p*, *q*, and *r*. Hence the two statement forms are logically equivalent.

p	q	r	$p \vee q$	$p \to r$	$q \to r$	$p \vee q \to r$	$(p \to r) \wedge (q \to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	T	T	T	T	T
T	F	F	T	F	T	F	F
F	T	T	T	T	T	T	T
F	T	F	T	T	F	F	F
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

$p \vee q \to r$ and $(p \to r) \wedge (q \to r)$
always have the same truth values,
so they are logically equivalent

Representation of If-Then As Or

Example 4 – *Application of the Equivalence between* $\sim p \vee q$ *and* $p \rightarrow q$

Rewrite the following statement in if-then form.

Either you get to work on time or you are fired.

Solution:

Let $\sim p$ be

You get to work on time.

and q be

You are fired.

Then the given statement is $\sim p \vee q$. Also p is

You do not get to work on time.

So the equivalent if-then version, $p \rightarrow q$, is

If you do not get to work on time, then you are fired.

The Negation of a Conditional Statement

Negation of a Conditional Statement

By definition, $p \rightarrow q$ is false if, and only if, its hypothesis, p, is true and its conclusion, q, is false. It follows that

The negation of "if p then q" is logically equivalent to "p and not q."

This can be restated symbolically as follows:

$$\sim(p \rightarrow q) \equiv p \wedge \sim q$$

Example 5 – *Negations of If-Then Statements*

Write negations for each of the following statements:

a. If my car is in the repair shop, then I cannot get to class.

b. If Sara lives in Athens, then she lives in Greece.

Solution:

a. My car is in the repair shop and I can get to class.

b. Sara lives in Athens and she does not live in Greece. (Sara might live in Athens, Georgia; Athens, Ohio; or Athens, Wisconsin.)

Contrapositive of a Conditional Statement

Contrapositive of a Conditional Statement

One of the most fundamental laws of logic is the equivalence between a conditional statement and its contrapositive.

- **Definition**

The **contrapositive** of a conditional statement of the form "If p then q" is

$$\text{If } \sim q \text{ then } \sim p.$$

Symbolically,

$$\text{The contrapositive of } p \rightarrow q \text{ is } \sim q \rightarrow \sim p.$$

A conditional statement is logically equivalent to its contrapositive.

Example 6 – *Writing the Contrapositive*

Write each of the following statements in its equivalent contrapositive form:

a. If Howard can swim across the lake, then Howard can swim to the island.

b. If today is Easter, then tomorrow is Monday.

Solution:

a. If Howard cannot swim to the island, then Howard cannot swim across the lake.

b. If tomorrow is not Monday, then today is not Easter.

The Converse and Inverse of a Conditional Statement

The Converse and Inverse of a Conditional Statement

The fact that a conditional statement and its contrapositive are logically equivalent is very important and has wide application. Two other variants of a conditional statement are *not* logically equivalent to the statement.

• Definition

Suppose a conditional statement of the form "If p then q" is given.

1. The **converse** is "If q then p."
2. The **inverse** is "If $\sim p$ then $\sim q$."

Symbolically,

$$\text{The converse of } p \to q \text{ is } q \to p,$$

and

$$\text{The inverse of } p \to q \text{ is } \sim p \to \sim q.$$

Example 7 – *Writing the Converse and the Inverse*

Write the converse and inverse of each of the following statements:

a. If Howard can swim across the lake, then Howard can swim to the island.

b. If today is Easter, then tomorrow is Monday.

Solution:

a. *Converse*: If Howard can swim to the island, then Howard can swim across the lake.

Inverse: If Howard cannot swim across the lake, then Howard cannot swim to the island.

Example 7 – *Solution*

cont'd

b. *Converse*: If tomorrow is Monday, then today is Easter.

Inverse: If today is not Easter, then tomorrow is not Monday.

The Converse and Inverse of a Conditional Statement

1. A conditional statement and its converse are *not* logically equivalent.

2. A conditional statement and its inverse are *not* logically equivalent.

3. The converse and the inverse of a conditional statement are logically equivalent to each other.

Only If and the Biconditional

Only If and the Biconditional

To say "*p* only if *q*" means that *p* can take place *only* if *q* takes place also. That is, if *q* does not take place, then *p* cannot take place.

Another way to say this is that if *p* occurs, then *q* must also occur (by the logical equivalence between a statement and its contrapositive).

• Definition

It p and q are statements,

p **only if** q means "if not q then not p,"

or, equivalently,

"if p then q."

Example 8 – *Converting Only If to If-Then*

Rewrite the following statement in if-then form in two ways, one of which is the contrapositive of the other.

John will break the world's record for the mile run only if he runs the mile in under four minutes.

Solution:

Version 1: If John does not run the mile in under four minutes, then he will not break the world's record.

Version 2: If John breaks the world's record, then he will have run the mile in under four minutes.

Only If and the Biconditional

• Definition

Given statement variables p and q, the **biconditional of p and q** is "p if, and only if, q" and is denoted $p \leftrightarrow q$. It is true if both p and q have the same truth values and is false if p and q have opposite truth values. The words *if and only if* are sometimes abbreviated **iff.**

The biconditional has the following truth table:

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Truth Table for $p \leftrightarrow q$

Only If and the Biconditional

In order of operations $\leftrightarrow$ is coequal with $\rightarrow$. As with $\wedge$ and $\vee$, the only way to indicate precedence between them is to use parentheses.

The full hierarchy of operations for the five logical operators is:

Order of Operations for Logical Operators

1.	$\sim$	Evaluate negations first.
2.	$\wedge, \vee$	Evaluate $\wedge$ and $\vee$ second. When both are present, parentheses may be needed.
3.	$\rightarrow, \leftrightarrow$	Evaluate $\rightarrow$ and $\leftrightarrow$ third. When both are present, parentheses may be needed.

Only If and the Biconditional

According to the separate definitions of *if* and *only if*, saying "*p* if, and only if, *q*" should mean the same as saying both "*p* if *q*" and "*p* only if *q*."

The following annotated truth table shows that this is the case:

p	q	$p \to q$	$q \to p$	$p \leftrightarrow q$	$(p \to q) \wedge (q \to p)$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

$p \leftrightarrow q$ and $(p \to q) \wedge (q \to p)$ always have the same truth values, so they are logically equivalent

Truth Table Showing that $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$

Example 9 – *If and Only If*

Rewrite the following statement as a conjunction of two if-then statements:

This computer program is correct if, and only if, it produces correct answers for all possible sets of input data.

Solution:

If this program is correct, then it produces the correct answers for all possible sets of input data; and if this program produces the correct answers for all possible sets of input data, then it is correct.

Necessary and Sufficient Conditions

Necessary and Sufficient Conditions

The phrases *necessary condition* and *sufficient condition*, as used in formal English, correspond exactly to their definitions in logic.

• Definition

If r and s are statements:

r is a **sufficient condition** for s means "if r then s."

r is a **necessary condition** for s means "if not r then not s."

In other words, to say "*r* is a sufficient condition for *s*" means that the occurrence of *r* is *sufficient* to guarantee the occurrence of *s*.

Necessary and Sufficient Conditions

On the other hand, to say "*r* is a necessary condition for *s*" means that if *r* does not occur, then *s* cannot occur either: The occurrence of *r* is necessary to obtain the occurrence of *s*. Note that because of the equivalence between a statement and its contrapositive,

r is a necessary condition for s also means "if s then r."

Consequently,

r is a necessary and sufficient condition for s means "r if, and only if, s."

Example 10 – *Interpreting Necessary and Sufficient Conditions*

Consider the statement "If John is eligible to vote, then he is at least 18 years old."

The truth of the condition "John is eligible to vote" is *sufficient* to ensure the truth of the condition "John is at least 18 years old."

In addition, the condition "John is at least 18 years old" is *necessary* for the condition "John is eligible to vote" to be true.

If John were younger than 18, then he would not be eligible to vote.

SECTION 2.3

Valid and Invalid Arguments

Valid and Invalid Arguments

In mathematics and logic an argument is not a dispute. It is a sequence of statements ending in a conclusion.

How to determine whether an argument is valid — that is, whether the conclusion follows *necessarily* from the preceding statements.

This determination depends only on the form of an argument, not on its content.

Valid and Invalid Arguments

For example, the argument

If Socrates is a man, then Socrates is mortal.
Socrates is a man.
$\therefore$ Socrates is mortal.

has the abstract form

If p then q
p
$\therefore q$

p and q as variables for which statements may be substituted.

An argument form is called *valid* if, and only if, whenever statements are substituted that make all the premises true, the conclusion is also true.

Valid and Invalid Arguments

• **Definition**

An **argument** is a sequence of statements, and an **argument form** is a sequence of statement forms. All statements in an argument and all statement forms in an argument form, except for the final one, are called **premises** (or **assumptions** or **hypotheses**). The final statement or statement form is called the **conclusion.** The symbol ∴, which is read "therefore," is normally placed just before the conclusion.

To say that an *argument form* is **valid** means that no matter what particular statements are substituted for the statement variables in its premises, if the resulting premises are all true, then the conclusion is also true. To say that an *argument* is **valid** means that its form is valid.

the truth of the premises.

Testing validity of argument form

1. Identify premises and conclusion of argument form.

2. Construct a truth table showing truth values of all premises and conclusion, under all possible truth values for variables.

3. A row of the truth table in which all the premises are true is called a **critical row**.

- If there is a critical row in which conclusion is false, then it is possible for an argument of the given form to have true premises and a false conclusion, and so the argument form is invalid.
- If conclusion in *every* critical row is true, then argument form is valid.

Example 1 – *Determining Validity or Invalidity* cont'd

$$p \rightarrow q \vee \sim r$$
$$q \rightarrow p \wedge r$$
$$\therefore \; p \rightarrow r$$

						premises		conclusion
p	q	r	$\sim r$	$q \vee \sim r$	$p \wedge r$	$p \rightarrow q \vee \sim r$	$q \rightarrow p \wedge r$	$p \rightarrow r$
T	T	T	F	T	T	T	T	T
T	T	F	T	T	F	T	F	
T	F	T	F	F	T	F	T	
T	F	F	T	T	F	T	T	F
F	T	T	F	T	F	T	F	
F	T	F	T	T	F	T	F	
F	F	T	F	F	F	T	T	T
F	F	F	T	T	F	T	T	T

This row shows that an argument of this form can have true premises and a false conclusion. Hence this form of argument is invalid.

S

there is one situation (row 4) where the premises are true and the conclusion is false.

Modus Ponens and Modus Tollens

Modus Ponens

An argument form consisting of two premises and a conclusion is called a **syllogism.**

The first and second premises are called the **major premise** and **minor premise**, respectively.

The most famous form of syllogism in logic is called **modus ponens** with the following form:

If *p* then *q*.

p

$\therefore$ *q*

Modus Ponens

It is instructive to prove that modus ponens is a valid form of argument, if for no other reason than to confirm the agreement between the formal definition of validity and the intuitive concept.

To do so, we construct a truth table for the premises and conclusion.

		premises		conclusion	
p	q	$p \to q$	p	q	
T	T	T	T	T	← critical row
T	F	F	T		
F	T	T	F		
F	F	T	F		

Modus Tollens

Now consider another valid argument form called **modus tollens**. It has the following form:

If p then q.

$\sim q$

$\therefore \sim p$

Example 2 – *Recognizing Modus Ponens and Modus Tollens*

Use modus ponens or modus tollens to fill in the blanks of the following arguments so that they become valid inferences.

a. If there are more pigeons than there are pigeonholes, then at least two pigeons roost in the same hole.
There are more pigeons than there are pigeonholes.
∴ ____.

b. If 870,232 is divisible by 6, then it is divisible by 3.
870,232 is not divisible by 3.
∴ ____.

Example 2 – *Solution*

a. At least two pigeons roost in the same hole. by modus ponens

b. 870,232 is not divisible by 6. by modus tollens

Additional Valid Argument Forms: Rules of Inference

Additional Valid Argument Forms: Rules of Inference

A **rule of inference** is a form of argument that is valid. Thus modus ponens and modus tollens are both rules of inference.

The following are additional examples of rules of inference that are frequently used in deductive reasoning.

Example 3 – *Generalization*

The following argument forms are valid:

a. p **b.** q

• $p \vee q$ • $p \vee q$

These argument forms are used for making generalizations. For instance, according to the first, if p is true, then, more generally, "p or q" is true for *any* other statement q.

As an example, suppose you are given the job of counting the upperclassmen at your school. You ask what class Anton is in and are told he is a junior.

Example 3 – *Generalization*

cont'd

You reason as follows:

Anton is a junior.

- (more generally) Anton is a junior or Anton is a senior.

Knowing that upperclassman means junior or senior, you add Anton to your list.

Example 4 – *Specialization*

The following argument forms are valid:

a. $p \wedge q$
$\therefore\ p$

b. $p \wedge q$
$\therefore\ q$

These argument forms are used for specializing. When classifying objects according to some property, you often know much more about them than whether they do or do not have that property.

When this happens, you discard extraneous information as you concentrate on the particular property of interest.

Example 4 – *Specialization*

cont'd

For instance, suppose you are looking for a person who knows graph algorithms to work with you on a project. You discover that Ana knows both numerical analysis and graph algorithms. You reason as follows:

Ana knows numerical analysis and Ana knows graph algorithms.

- (in particular) Ana knows graph algorithms.

Accordingly, you invite her to work with you on your project.

Additional Valid Argument Forms: Rules of Inference

Both generalization and specialization are used frequently in mathematics to tailor facts to fit into hypotheses of known theorems in order to draw further conclusions.

Elimination, transitivity, and proof by division into cases are also widely used tools.

Example 5 – *Elimination*

The following argument forms are valid:

a. $p \vee q$

$\sim q$

$\therefore\ p$

b. $p \vee q$

$\sim p$

$\therefore\ q$

These argument forms say that when you have only two possibilities and you can rule one out, the other must be the case. For instance, suppose you know that for a particular number x,

$$x - 3 = 0 \quad \text{or} \quad x + 2 = 0.$$

Example 5 – *Elimination*

cont'd

If you also know that x is not negative, then $x \neq -2$, so

$$x + 2 \neq 0.$$

By elimination, you can then conclude that

$$\therefore \bullet \quad x - 3 = 0.$$

Example 6 – *Transitivity*

The following argument form is valid:

$$p \rightarrow q$$

$$q \rightarrow r$$

$$\bullet \quad p \rightarrow r$$

Many arguments in mathematics contain chains of if-then statements.

From the fact that one statement implies a second and the second implies a third, you can conclude that the first statement implies the third.

Example 6 – *Transitivity*

cont'd

Here is an example:

If 18,486 is divisible by 18, then 18,486 is divisible by 9.

If 18,486 is divisible by 9, then the sum of the digits of 18,486 is divisible by 9.

- If 18,486 is divisible by 18, then the sum of the digits of 18,486 is divisible by 9.

Example 7 – *Proof by Division into Cases*

The following argument form is valid:

$$\begin{array}{ll} & p \vee q \\ & p \to r \\ & q \to r \\ \therefore & r \end{array}$$

It often happens that you know one thing or another is true. If you can show that in either case a certain conclusion follows, then this conclusion must also be true.

For instance, suppose you know that x is a particular nonzero real number.

Example 7 – *Proof by Division into Cases* cont'd

The trichotomy property of the real numbers says that any number is positive, negative, or zero. Thus (by elimination) you know that x is positive or x is negative.

You can deduce that $x^2 > 0$ by arguing as follows:

x is positive or x is negative.

If x is positive, then $x^2 > 0$.

If x is negative, then $x^2 > 0$.

- $x^2 > 0$.

Additional Valid Argument Forms: Rules of Inference

The rules of valid inference are used constantly in problem solving. Here is an example from everyday life.

Example 8 – *Application: A More Complex Deduction*

You are about to leave for school in the morning and discover that you don't have your glasses. You know the following statements are true:

a. If I was reading the newspaper in the kitchen, then my glasses are on the kitchen table.

b. If my glasses are on the kitchen table, then I saw them at breakfast.

c. I did not see my glasses at breakfast.

d. I was reading the newspaper in the living room or I was reading the newspaper in the kitchen.

e. If I was reading the newspaper in the living room then my glasses are on the coffee table.

Where are the glasses?

Example 8 – *Application: A More Complex Deduction* cont'd

Solution:

Let *RK* = I was reading the newspaper in the kitchen.

GK = My glasses are on the kitchen table.

SB = I saw my glasses at breakfast.

RL = I was reading the newspaper in the living room.

GC = My glasses are on the coffee table.

Example 8 – *Solution*

cont'd

Here is a sequence of steps you might use to reach the answer, together with the rules of inference that allow you to draw the conclusion of each step:

1.

$RK \rightarrow GK$ by (a)

$GK \rightarrow SB$ by (d)

- $RK \rightarrow SB$ by transitivity

2.

$RK \rightarrow SB$ by the conclusion of (1)

$\sim SB$ by (c)

- $\sim RK$ by modus tollens

Example 8 – *Solution*

cont'd

3. $RL \lor RK$ by (d)

$\sim RK$ by the conclusion of (2)

- RL by elimination

4.

$RL \to GC$ by (e)

RL by the conclusion of (3)

- GC by modus ponens

Thus the glasses are on the coffee table.

Fallacies

Fallacies

A **fallacy** is an error in reasoning that results in an invalid argument. Three common fallacies are **using ambiguous premises,** and treating them as if they were unambiguous, **circular reasoning** (assuming what is to be proved without having derived it from the premises), and **jumping to a conclusion** (without adequate grounds).

In this section we discuss two other fallacies, called *converse error* and *inverse error*, which give rise to arguments that superficially resemble those that are valid by modus ponens and modus tollens but are not, in fact, valid.

Fallacies

For an argument to be valid, every argument of the same form whose premises are all true must have a true conclusion. It follows that for an argument to be invalid means that there is an argument of that form whose premises are all true and whose conclusion is false.

Example 9 – *Converse Error*

Show that the following argument is invalid:

If Zeke is a cheater, then Zeke sits in the back row.

Zeke sits in the back row.

∴ Zeke is a cheater.

Solution:

Many people recognize the invalidity of the above argument intuitively, reasoning something like this:

The first premise gives information about Zeke *if* it is known he is a cheater. It doesn't give any information about him if it is not already known that he is a cheater.

Example 9 – *Solution*

cont'd

One can certainly imagine a person who is not a cheater but happens to sit in the back row. Then if that person's name is substituted for Zeke, the first premise is true by default and the second premise is also true but the conclusion is false.

The general form of the previous argument is as follows:

$$p \to q$$
$$q$$
$$p$$

Fallacies

The fallacy underlying this invalid argument form is called the **converse error** because the conclusion of the argument would follow from the premises if the premise $p \rightarrow q$ were replaced by its converse.

Such a replacement is not allowed, however, because a conditional statement is not logically equivalent to its converse. Converse error is also known as the *fallacy of affirming the consequent*.

Another common error in reasoning is called the *inverse error*.

Example 10 – *Inverse Error*

Consider the following argument:

If interest rates are going up, stock market prices will go down.

Interest rates are not going up.

• Stock market prices will not go down.

Note that this argument has the following form:

$$p \to q$$

$$\sim p$$

$$\bullet\ \sim q$$

Example 10 – *Inverse Error*

cont'd

The fallacy underlying this invalid argument form is called the **inverse error** because the conclusion of the argument would follow from the premises if the premise $p \rightarrow q$ were replaced by its inverse.

Such a replacement is not allowed, however, because a conditional statement is not logically equivalent to its inverse. Inverse error is also known as the *fallacy of denying the antecedent*.

Example 11 – *A Valid Argument with a False Premise and a False Conclusion*

The argument below is valid by modus ponens. But its major premise is false, and so is its conclusion.

If John Lennon was a rock star, then John Lennon had red hair.

John Lennon was a rock star.

• John Lennon had red hair.

Example 12 – *An Invalid Argument with True Premises and a True Conclusion*

The argument below is invalid by the converse error, but it has a true conclusion.

If New York is a big city, then New York has tall buildings.

New York has tall buildings.

∴ New York is a big city.

Fallacies

• Definition

An argument is called **sound** if, and only if, it is valid *and* all its premises are true. An argument that is not sound is called **unsound**.

Contradictions and Valid Arguments

Contradictions and Valid Arguments

The concept of logical contradiction can be used to make inferences through a technique of reasoning called the *contradiction rule*. Suppose p is some statement whose truth you wish to deduce.

Contradiction Rule

If you can show that the supposition that statement p is false leads logically to a contradiction, then you can conclude that p is true.

Example 13 – *Contradiction Rule*

Show that the following argument form is valid:

$$\sim p \to c, \text{ where } c \text{ is a contradiction}$$
$$\therefore\ p$$

Solution:

Construct a truth table for the premise and the conclusion of this argument.

			premises	conclusion
p	$\sim p$	c	$\sim p \to c$	p
T	F	F	T	T
F	T	F	F	

There is only one critical row in which the premise is true, and in this row the conclusion is also true. Hence this form of argument is valid.

Contradictions and Valid Arguments

The contradiction rule is the logical heart of the method of proof by contradiction.

A slight variation also provides the basis for solving many logical puzzles by eliminating contradictory answers: *If an assumption leads to a contradiction, then that assumption must be false.*

Summary of Rules of Inference

Modus Ponens	$p \to q$ p • q	**Elimination**	**a.** $p \lor q$ $\sim q$ • p **b.** $p \lor q$ $\sim p$ • q
Modus Tollens	$p \to q$ $\sim q$ • $\sim p$	**Transitivity**	$p \to q$ $q \to r$ • $p \to r$
Generalization	**a.** p • $p \lor q$ **b.** q • $p \lor q$	**Proof by Division into Cases**	$p \lor q$ $p \to r$ $q \to r$ • r
Specialization	**a.** $p \land q$ • p **b.** $p \land q$ • q		
Conjunction	p q • $p \land q$	**Contradiction Rule**	$\sim p \to c$ • p

Valid Argument Forms

Table 2.3.1