Computer Language Theory

Chapter 4: Decidability

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Limitations of Algorithmic Solvability

- In this chapter we investigate the power of algorithms to solve problems
  - Some can be solved algorithmically and some <u>cannot</u>
- Why we study unsolvability
  - Useful because then can realize that searching for an algorithmic solution is a waste of time
    - Perhaps the problem can be simplified
  - Gain an perspective on computability and its limits
  - In my view also related to complexity (Chapter 7)
    - First we study whether there is an algorithmic solution and then we study whether there is an "efficient" (polynomial-time) one

Chapter 4.1

Decidable Languages

Decidable Languages

- We start with problems that are decidable
  - We first look at problems concerning regular languages and then those for context-free languages

Decidable Problems for Regular Languages

- We give algorithms for testing whether a finite automaton accepts a string, whether the language of a finite automaton is empty, and whether two finite automata are equivalent
- We represent the problems by languages (not FAs)
  - Let $A_{DFA}=\{(B, w) \mid B \text{ is a DFA that accepts string } w\}$
  - The problem of testing whether a DFA B accepts a specific input w is the same as testing whether (B,w) is a member of the language A_{DFA}.
  - Showing that the language is decidable is the same thing as showing that the computational problem is decidable
  - So do you understand what A_{DFA} represents? If you had to list the elements of A_{DFA} what would they be?

A_{DFA} is a Decidable Language

- Theorem: A_{DFA} is a decidable language
- Proof Idea: Present a TM M that decides A_{DFA}
  - M = On input (B,w), where B is a DFA and w is a string:
    1. Simulate B on input w
    2. If the simulation ends in an accept state, then accept; else reject

Outline of Proof

- Must take B as input, described as a string, and then simulate it
  - This means the algorithm for simulating any DFA must be embodied in the TM's state transitions
  - Think about this. Given a current state and input symbol, scan the tape for the encoded transition function and then use that info to determine new state
- The actual proof would describe how a TM simulates a DFA
  - Can assume B is represented by its 5 components and then we have w
    - Note that the TM must be able to handle <u>any</u> DFA, not just this one
  - Keep track of current state and position in w by writing on the tape
    - Initially current state is q0 and current position is leftmost symbol of w
  - The states and position are updated using the transition function δ
    - TM M's δ not the same as DFA B's δ
  - When M finishes processing, accept if in an accept state; else reject. The implementation will make it clear that will complete in finite time.

A_{NFA} is a Decidable Language

- Proof Idea:
  - Because we have proven decidability for DFAs, all we need to do is convert the NFA to a DFA.
    - N = On input (B,w) where B is an NFA and w is a string
      1. Convert NFA B to an equivalent DFA C, using the procedure for conversion given in Theorem 1.39
      2. Run TM M on input (C,w) using the theorem we just proved
      3. If M accepts, then accept; else reject
  - Running TM M in step 2 means incorporating M into the design of N as a subroutine
  - Note that these proofs allow the TM to be described at the highest of the 3 levels we discussed in Chapter 3 (and even then, without most of the details!).

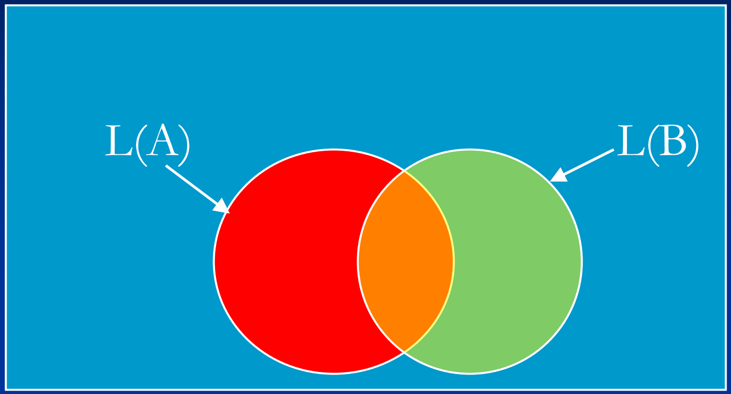
Computing whether a DFA accepts any String

- E_{DFA} = {<A>| A is a DFA and L(A) = ∅) is a decidable language
- Proof:
  - A DFA accepts some string iff it is possible to reach the accept state from the start state. How can we check this?
  - We can use a marking algorithm similar to the one used in Chapter 3.
  - T = On input (A) where A is a DFA:
    1. Mark the start state of A
    2. Repeat until no new states get marked:
       3. Mark any state that has a transition coming into it from any state already marked
    4. If no accept state is marked, accept; otherwise reject
  - In my opinion this proof is clearer than most of the previous ones because the pseudo-code above specifies enough details to make it clear how to implement it

EQ_{DFA} is a Decidable Language

- EQ_{DFA} = {(A,B) | A and B are DFAs and L(A)=L(B)}
- Proof idea
  - Construct a DFA C from A and B, where C accepts only those strings accepted by either A or B but not both (symmetric difference)
    - If A and B accept the same language, then C will accept nothing and we can use the previous proof (for E_{DFA}) to check for this.
    - So, the proof is:
      - F = On input (A,B) where A and B are DFAs:
        1. Construct DFA C that is the symmetric difference of A and B (details on how to do this on next slide)
        2. Run TM T from the proof from last slide on input (C)
        3. If T accepts (sym. diff=∅) then accept. If T rejects then reject

How to Construct C



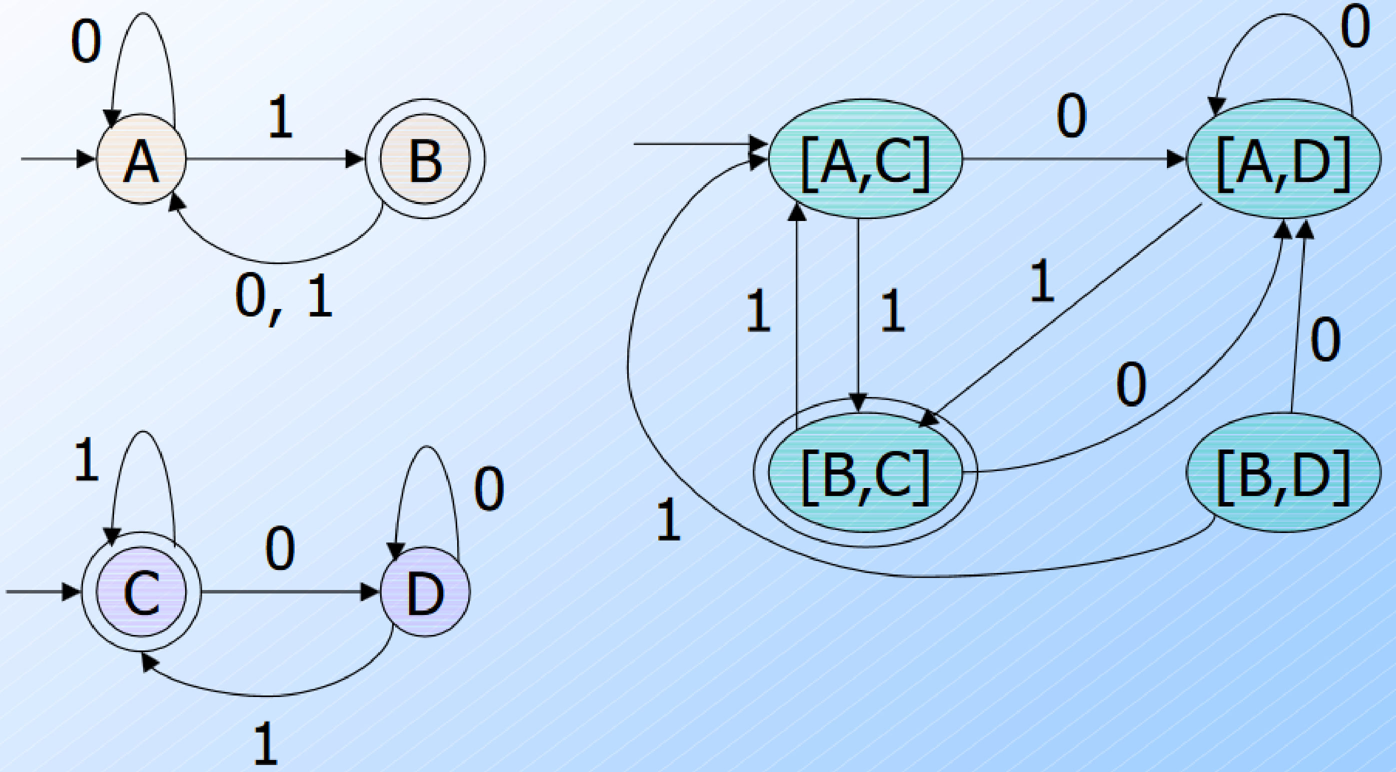
Complement symbol

- $L(C) = (L(A) \cap L(B)') \cup (L(A)' \cap L(B))$
  - We used proofs by construction that regular languages are closed under $\cup$, $\cap$, and complement
  - We can use those constructions to construct a FA that accepts L(C)
    - Wait a minute! The book is quite cavalier! We never proved regular languages are closed under $\cap$

Regular Languages Closed under Intersection

- If L and M are regular languages, then so is L ∩ M
- Proof: Let A and B be DFAs whose regular languages are L and M, respectively
- Construct C, the "product automation" of A and B
  - More on this in a minute, but essentially C tracks the states in A and B (just like when we did the proof of union without using NFAs)
- Make the final states of C be the pairs consisting of final states of both A and B
  - In the union case we the final state any state with a final state in A or B

Example: Product DFA for Intersection



A_{CFG} is a Decidable Language

- Proof Idea:
  - For CFG G and string w want to determine whether G generates w. One idea is to use G to go through all derivations. This will not work, why?
    - Because this method a best will yield a TM that is a recognizer, not a decider. Can generate infinite strings and if not in the language, will never know it.
    - But since we know the length of w, we can exploit this. How?
    - A string w of length n will have a derivation that uses 2n-1 steps if the CFG is in Chomsky-Normal Form.
      - So first convert to Chomsky-Normal Form
      - Then list all derivations of length 2n-1 steps. If any generates w, then accept, else reject.
      - This is a variant of breadth first search, but instead of extended the depth 1 at a time we allow it to go 2n-1 at a time. As long as finite depth extension, we are okay

E_{CFG} is a Decidable Language

- How can you do this? What is the brute force approach?
  - Try all possible strings w. Will this work?
    - The number is not bounded, so this would not be decidable
    - Instead, think of this as a graph problem where you want to know if you can reach a string of terminals from the start state
    - Do you think it is easier to work forward or backwards?
    - Answer: backwards

E_{CFG} is a Decidable Language (cont)

- Proof Idea:
  - Can the start variable generate a string of terminals?
  - Determine for each variable if it can generate any string of terminals and if so, mark it
  - Keep working backwards so that if the right-side of any rule has only marked items, then mark the LHS
    - For example, if X→ YZ and Y and Z are marked, then mark X
    - If you mark S, then done; if nothing else to mark and S not marked, then reject
    - You start by marking all terminal symbols

EQ_{CFG} is not a Decidable Language

- We cannot reuse the reasoning to show that EQ_{DFA} is a decidable language since CFGs are not closed under complement and intersection
- As it turns out, EQ_{CFG} is not decidable!
- We will learn in Chapter 5 how to prove things undecidable

Every Context-Free Language is Decidable

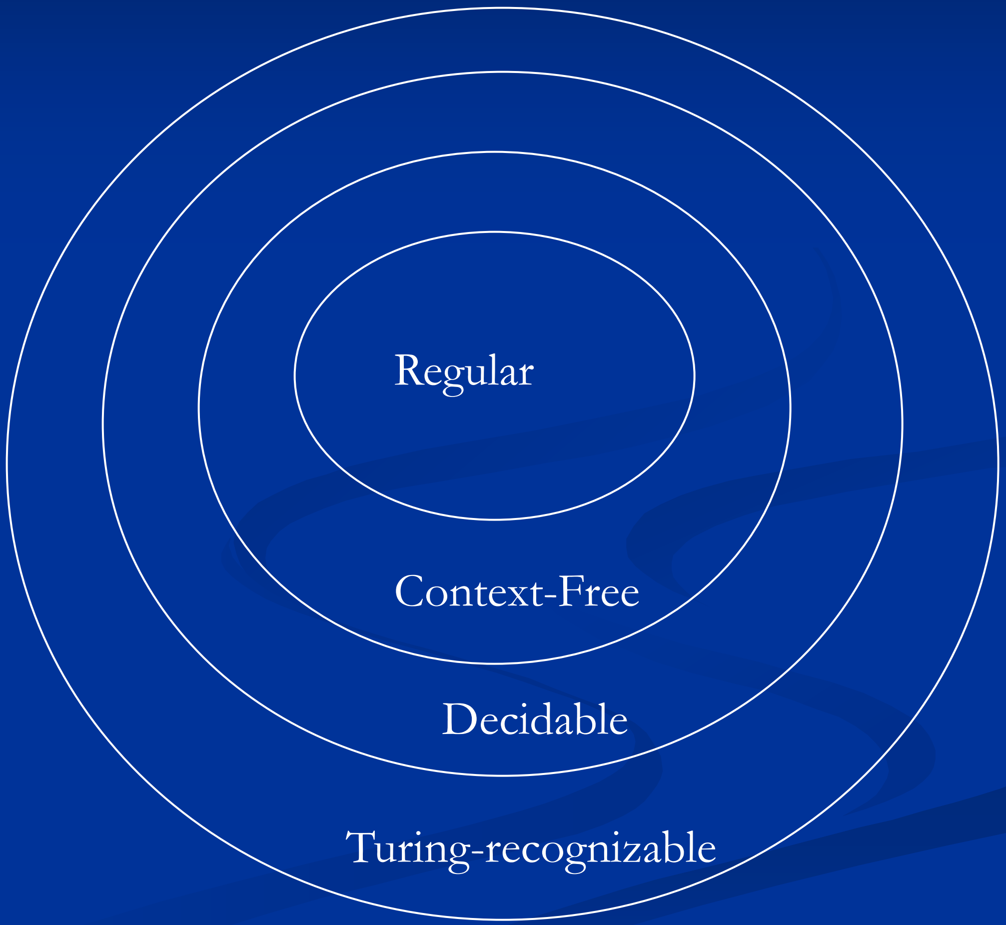
- Note that a few slides back we showed A_{CFG} is decidable.
- This is almost the same thing
- We want to know if A, which is a CFL, is decidable.
  - A will have some CFG G that generates it
  - When we proved that A_{CFG} is decidable, we constructed a TM S that would tell us if any CFG accepts a particular input w.
  - Now we use this TM and run it on input <G,w> and if it accepts, we accept, and if it rejects, we reject.
- This is so close to the prior proof it is confusing. It comes from the fact that a CFL is defined by a CFG.
- This leads us to the following picture of the hierarchy of languages

Hierarchy of Classes of Languages

We proved Regular $\subseteq$ Context-free since we can convert a FA into a CFG

We just proved that every Context-free language is decidable

From the definitions in Chapter 3 it is clear that every Decidable language is trivially Turing-recognizable. We hinted that not every Turing-recognizable language is Decidable. Next we prove that!



Chapter 4.2

The Halting Problem

The Halting Problem

- One of the most philosophically important theorems in the theory of computation
  - There is a specific problem that is algorithmically unsolvable.
  - In fact, ordinary/practical problems may be unsolvable
    - Software verification
      - Given a computer program and a precise specification of what the program is supposed to do (e.g., sort a list of numbers)
      - Come up with an algorithm to prove the program works as required
        - This cannot be done!
        - But wait, can't we prove a sorting algorithm works?
        - Note: the input has two parts: specification and task. The proof is not only to prove it works for a specific task, like sorting numbers.
- Our first undecidable problem:
  - Does a TM accept a given input string?
    - Note: we have shown that a CFL is decidable and a CFG can be simulated by a TM. This does not yield a contradiction. TMs are more expressive than CFGs.

Halting Problem II

- A_{TM} = {(M,w) | M is a TM and M accepts w}
- A_{TM} is undecidable
  - It can only be undecidable due to a loop of M on w.
  - If we could determine if it will loop forever, then could reject. Hence A_{TM} is often called the halting problem.
    - As we will show, it is impossible to determine if a TM will always halt (i.e., on every possible input).
  - Note that this is Turing recognizable:
    - Simulate M on input w and if it accept, then accept; if it ever rejects, then reject
  - We start with the diagonalization method

Diagonalization Method

- In 1873 mathematician Cantor was concerned with the problem of measuring the sizes of infinite sets.
  - How can we tell if one infinite set is bigger than another or if they are the same size?
    - We cannot use the counting method that we would use for finite sets. Example: how many even integers are there?
    - What is larger: the set of even integers or the set of all strings over {0,1} (which is the set of all integers)
  - Cantor observed that two finite sets have the same size if each element in one set can be paired with the element in the other
    - This can work for infinite sets

Function Property Definitions

- From basic discrete math (e.g., CS 1100)
  - Given a set A and B and a function *f* from A to B
    - *f* is one-to-one if it never maps two elements in A to the same element in B
      - The function *add-two* is one-to-one whereas *absolute-value* is not
    - *f* is onto if every item in B is reached from some value in a (i.e., f(a) = b for every b ∈ B).
      - For example, if A and B are the set of integers, then *add-two* is onto but if A and B are the positive integers, then it is not onto since b = 1 is never hit.
    - A function that is one-to-one and onto has a (one-to-one) correspondence
      - This allows all items in each set to be paired

An Example of Pairing Set Items

- Let N be the set of natural numbers {1, 2, 3, …} and let E be the set of even natural numbers {2, 4, 6, …}.
- Using Cantor's definition of size we can see that N and E have the same size.
  - The correspondence f from N to E is f(n) = 2n.
- This may seem bizarre since E is a proper subset of N, but it is possible to pair all items, since f(n) is a 1:1 correspondence, so we say they are the same size.
- Definition:
  - A set is *countable* if either it is finite or it has the same size as N, the set of natural numbers

Example: Rational Numbers

- Let Q = {m/n: m,n ∈ N}, the set of positive Rational Numbers
- Q seems much larger than N, but according to our definition, they are the same size.
  - Here is the 1:1 correspondence between Q and N
  - We need to list all of the elements of Q and then label the first with 1, the second with 2, etc.
    - We need to make sure each element in Q is listed only once

Correspondence between N and Q

- To get our list, we make an infinite matrix containing all the positive rational numbers.
  - Bad way is to make the list by going row-to-row. Since 1st row is infinite, would never get to the second row
  - Instead use the diagonals, not adding the values that are equivalent
    - So the order is 1/1, 2/1, ½, 3/1, 1/3, …
  - This yields a correspondence between Q and N
    - That is, N=1 corresponds to 1/1, N=2 corresponds to 2/1, N=3 corresponds to ½ etc.

1/1	1/2	1/3	1/4	1/5
2/1	2/2	2/3	2/4	2/5
3/1	3/2	3/3	3/4	3/5
4/1	4/2	4/3	4/4	4/5
5/1	5/2	5/3	5/4	5/5

Theorem: R is Uncountable

- A real number is one that has a decimal representation and R is set of Real Numbers
  - Includes those that cannot be represented with a finite number of digits, like Pi and square root of 2
- Will show that there can be no pairing of elements between R and N
  - Will find some x that is always not in the pairings and thus a proof by contradiction

Finding a New Value x

- To the right is an example mapping
  - Assume that it is complete
- I now describe a method that will be guaranteed to generate a value x not already in the infinite list
- Generate x to be a real number between 0 and 1 as follows
  - To ensure that x ≠ f(1), pick a digit not equal to the first digit after the decimal point. Any value not equal to 1 will work. Pick 4 so we have .4
  - To x ≠ f(2), pick a digit not equal to the second digit. Any value not equal to 5 will work. Pick 6. We have .46
  - Continue, choosing values along the "diagonal" of digits (i.e., if we took the f(n) column and put one digit in each column of a new table).
- When done, we are guaranteed to have a value x not already in the list since it differs in at least one position with every other number in the list.

n	f(n)
1	3.14159…
2	55.5555…
3	0.12345…
4	0.500000
.	.

Implications

- The theorem we just proved about R being uncountable has an important application in the theory of computation
  - It shows that some languages are not decidable or even Turing-recognizable, because there are uncountably many languages yet only countably many Turing Machines.
    - Because each Turing machine can recognize a single language and there are more languages than Turing machines, some languages are not recognized by any Turing machine.
      - Corollary: some languages are not Turing-recognizable

Some Languages are Not Turing-recognizable I

- The set of all strings $\sum^*$ is countable
  - A finite number of strings of each length, so we can list them by increasing length and hence they are countable
- The set of all Turing Machines M is countable since each TM M has an encoding into a string <M>
  - Order by length and omit strings that do not represent valid TM’s and we have a countable list of Turing Machines

Some Languages are Not Turing-recognizable II

- The set of all languages L over ∑ is uncountable
  - Recall a language is made up of a set of strings, so different from what we just counted on last slide.
  - Each language is represented by an infinite binary sequence B, where each position in the sequence corresponds to a string
    - Assume $\Sigma^* = \{s_1, s_2, s_3 \ldots\}$. We can encode any language as a characteristic binary sequence, where the bit indicates whether the corresponding s_i is a member of the language. Thus, there is a 1:1 mapping.
    - The set of all infinite binary sequences B is uncountable
    - Can prove uncountable using same proof used to prove real numbers not countable
  - L is uncountable because it has a correspondence with B
  - Since B is uncountable and L and B are of equal size, L is uncountable
- So set of TMs is countable and the set of languages is not
  - Means we cannot put set of languages into a correspondence with set of TMs.
  - Therefore some languages do not have a corresponding Turing machine
  - Thus some languages not Turing-Recognizable

Common Sense Explanation

- Comparing languages, a potentially infinite set of strings, versus number of strings
- Each language is represented by a sequence of infinite length whereas each individual string is of finite (but unbounded) length
- String is to Language as Natural number is to Real Number

Halting Problem is Undecidable

- Prove that halting problem is undecidable
  - We started this a while ago …
  - Let A_{TM} = {<M,w> | M is a TM and accepts w}
- Proof Technique:
  - Assume A_{TM} is decidable and obtain a contradiction
  - A diagonalization proof

Proof: Halting Problem is Undecidable

- Assume A_{TM} is decidable
- Let H be a decider for A_{TM}
  - On input <M,w>, where M is a TM and w is a string, H halts and accepts if M accepts w; otherwise it rejects
- Construct a TM D using H as a subroutine
  - D calls H to determine what M does when input string is its own description <M>.
    - Like running a C++ program where input is the program represented as a string
  - D then outputs the opposite of H's answer
  - D(<M>) accepts if M does not accept <M> and rejects if M accepts <M>
- Now run D on its own description
  - D(<D>) = accept if D does not accept <D> and reject if D accepts <D>
  - No matter what D does it is forced to do the opposite, which is a contradiction. Thus, neither TM D or TM H can exist. *See next slide.*

The Diagonalization Proof

	<M1>	<M2>	<M3>	<M4>	…	<D>	
M1	Accept	Reject	Accept	Reject	…	Accept	
M2	Accept	Accept	Accept	Accept	…	Accept	
M3	Reject	Reject	Reject	Reject	…	Reject	
M4	Accept	Accept	Reject	Reject	…	Accept	
.							
D	Reject	Reject	Accept	Accept	…	?	
.							

The TM D must invert the value on the diagonal. It can do this for <M1>, <M2>, etc, but not for <D>. If the entry for D(<D>) was accept then it needs to be reject, and if it was reject then it needs to be accept. Contradiction. Similar to proof that Real numbers not countable.

A More Satisfying Proof for CS Majors

- The last proof uses some mathematical tricks and is not very intuitive
- Computer programs appear more concrete to most of us
- The halting problem naturally is about programs and infinite loops
- After teaching this many times, I developed a proof that most of you will find less arbitrary since it focuses programs and infinite loops.
  - But it is nonetheless follows the same steps as the prior proof

Slightly more Concrete Version

- You write a program, halts(P, X) in C^{++} that takes as input any C^{++} program, P, and the input to that program, X
  - Your program halts(P, X) analyzes P and returns "yes" if P will halt on X and "no" if P will not halt on X
- You now write a short procedure foo(X):

  foo(X) {a: if halts(X,X) then goto a; else halt}

  This program does not halt if P halts on X (infinite loop via goto) and it does if P does not halt on X
- Does foo(foo) halt?
  - It halts if and only if halts(foo,foo) returns no
    - It halts if and only if it does not halt. Contradiction.
- Thus we have proven that you cannot write a program to determine if any arbitrary program will halt or loop

What does this mean?

- Recall what was said earlier
  - The halting problem is not some contrived problem
  - The halting problem asks whether we can tell if some TM M will accept an input string
  - We are asking if the language below is decidable
    - A_{TM} = {(M,w) | M is a TM and M accepts w}
  - It is not decidable
    - But as I keep emphasizing, M is an input variable too!
      - Of course, some algorithms are decidable, like sorting algorithms
  - Halting problem is Turing-recognizable (we discussed this)
    - Simulate the TM on w and if it accepts/rejects, then accept/reject.
  - The halting problem is special because it gets at the heart of the matter (it is related to A_{TM} in general)

Co-Turing Recognizable

- A language is co-Turing recognizable if it is the complement of a Turing-recognizable language
- Theorem: A language is decidable if and only if it is Turing-recognizable and co-Turing-recognizable
  - Why? To be Turing-recognizable, we must accept in finite time. If we don’t accept, we may reject or loop (it which case it is not decidable).
    - Since we can invert any “question” by taking the complement, taking the complement flips the accept and reject answers. Thus, if we invert the question and it is Turing-recognizable, then that means that we would get the answer to the original reject question in finite time.

More Formal Proof

- Theorem: A language is decidable iff it is Turing-recognizable and co-Turing-recognizable
- Proof (2 directions)
  - Forward direction easy. If it is decidable, then both it and its complement are Turing-recognizable
  - Other direction:
    - Assume A and A' are Turing-recognizable and let M1 recognize A and M2 recognize A'
    - The following TM will decide A
    - M = On input w
      1. Run both M1 and M2 on input w in parallel
      2. If M1 accepts, accept; if M2 accepts, then reject
    - Every string is in either A or A' so every string w must be accepted by either M1 or M2. Because M halts whenever M1 or M2 accepts, M always halts and so is a decider.
    - Furthermore, it accepts all strings in A and rejects all not in A, so M is also a decider for A and thus A is decidable

Implication

- For any undecidable language, either the language or its complement is not Turing-recognizable

Complement of A_{TM} is not Turing-recognizable

- A_{TM}' is not Turing-recognizable
- Proof:
  - We know that A_{TM} is Turing-recognizable but not decidable
  - If A_{TM}' were also Turing-recognizable, then A_{TM} would be decidable, which it is not
  - Thus A_{TM}' is not Turing-recognizable
- This should not be too surprising.
  - It is harder to determine that something is not in the language

Computer Language Theory

Chapter 5: Reducibility

Due to time constraints we are only going to cover the first 3 pages of this chapter. However, we cover the notion of reducibility in depth when we cover Chapter 7.

What is Reducibility?

- A reduction is a way of converting one problem to another such that the solution to the second can be used to solve the first
  - We say that problem A is reducible to problem B
  - Example: finding your way around NY City is reducible to the problem of finding and reading a map
  - If A reduces to B, what can we say about the relative difficulty of problem A and B?
    - A can be no harder than B since the solution to B solves A
    - A could be easier (the reduction is “inefficient” in a sense)
    - In example above, A is easier than B since B can solve any routing problem

Practice on Reducibility

- In our previous class work, did we reduce NFAs to DFAs or DFAs to NFAs?
  - We reduced NFAs to DFAs
    - We showed that an NFA can be reduced (i.e., converted) to a DFA via a set of simple steps
    - NFA can not be any more powerful than a DFA
    - Based only on the reduction, NFA could be less powerful
    - But since we know this is not possible, since an DFA is a degenerate form of an NFA, we showed they have the same expressive power

How Reducibility is used to Prove Languages Undecidable

- If A is reducible to B and B is decidable, what can we say?
  - A is decidable (since A can only be "easier")
  - Also, B, which is decidable, can be used to solve A
- If A is reducible to B and A is decidable, what can we say?
  - Nothing– B may not be decidable (so this is not useful for us)
- If A is undecidable and reducible to B, then what can we say about B?
  - B must be undecidable (B can only be harder than A)
  - This is the most useful part for Chapter 5, since this is how we can prove a language undecidable
    - We can leverage past proofs and not start from scratch
- To show something undecidable, show an undecidable problem can be reduced to it.

Example: Prove $HALT_{TM}$ is Undecidable I

- Need to reduce A_{TM} to $HALT_{TM}$, where A_{TM} already proven to be undecidable
  - Can use $HALT_{TM}$ to solve A_{TM}
- Proof by contradiction
  - Assume $HALT_{TM}$ is decidable and show this implies A_{TM} is decidable
    - Assume TM R that decides $HALT_{TM}$
    - Use R to construct S a TM that decides A_{TM}
    - Pretend you are S and need to decide A_{TM} so if given input <M, w> must output accept if M accepts w and reject if M loops on w or rejects w.
      - First try: simulate M on w and if it accepts then accept and if rejects then reject. But in trouble if it loops.
      - This is bad because we need to be a decider

Example: Prove $HALT_{TM}$ is Undecidable II

- Instead, use assumption that have TM R that decides $HALT_{TM}$
- Now can test if M halts on w
  - If R indicates that M does halt on w, you can use the simulation and output the same answer
  - If R indicates that M does not halt, then reject since infinite looping on w means it will never accept
  - The formal solution on next slide
  - We already discussed this case when we informally discussed how the halting problem is related to A_{TM}

Solution: $HALT_{TM}$ is Undecidable

- Assume TM R decides $HALT_{TM}$
- Construct TM S to decide A_{TM} as follows

S = “On input <M, w>, an encoding of a TM M and a string w:

1. Run TM R on input <M, w>
2. If R rejects (doesn’t halt), *reject*
3. If R accepts, simulate M on w until it halts
4. If M has accepted, *accept*; If M has rejected, *reject*”