Discrete Mathematics: An Introduction

This is a preliminary draft of a new textbook being developed for teaching discrete mathematics.

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Chapter 1: Combinatorics

This chapter introduces the major elements of combinatorics. While combinatorics is about more than just counting, counting (sometimes called enumerative combinatorics) is the focus of this chapter. Counting might seem very simple and a topic unfit for a college level class, but in this chapter we will see that counting is not always a straightforward task. The counting that is covered in this chapter does not involve counting actual objects, but rather "possibilities." For example, a typical combinatorics question we encounter is: "How many ways can you configure an automobile if there are four possible colors, three different safety options, and a sunroof is optional?"

Why do we care about combinatorics? One reason is that combinatorics is required in order to understand and apply probability theory—and probability theory has many important and practical applications. The connection between counting and probability will be made clear when we introduce probability in Chapter 2.

1.1 Counting and How to Count

While basic counting is often mastered by 4-year old children, there are some aspects that are worth commenting on. First, our counting questions are really about sets, a basic construct of discrete mathematics. If one asks, "What are all the integers between 4 and 10", the answer can be expressed by the set {5, 6, 7, 8, 9} and similarly, if one asks for all of the male students in a class, the answer can also be represented by a set. Note that in this case, as in many instances, we care not only about the number of objects in the set (i.e., the cardinality of the set), but also about the "names" of the objects in the set. That is, for these counting questions we sometimes want to enumerate the items in the set, not just count them.

The next key issue involves how to count. This topic, although seemingly simple, will be covered in the remainder of the chapter. But in this context we simply mean how can we methodically enumerate a set of items. If we ask you to enumerate the students in a class, it is clear how you might do this. You would simply create a list of the students. Of course, since we are interested in a set the actual ordering is not important, but most people would order the list alphabetically or by the seating position of each student in class, because these are convenient ways to ensure than no student is missed or counted multiple times.

In some cases this linear ordering to objects is not the most useful representation. For example, suppose you are asked to select a pair of men's jeans and you know that four styles are available (standard fit, loose fit, boot fit, and slim fit) and each style comes in two colors (blue or black). How could you count, or enumerate, the set of possible jean configurations? You could generate a list, which would not be very hard with only two "features", but this method would not work very well for more complex problems. The best way to enumerate the configurations is with a tree structure. This type of structure is very common in Computer Science and actually looks like an upside down tree, with the root at the top and the leaves at the bottom. Each level in the tree corresponds to a different "feature" or "variable" and each branch corresponds to a choice associated with

that feature or variable. The tree corresponding to this problem is shown below in Figure 1-1, with the first branch corresponding to jean style and the second branch corresponding to jean color. Note that there are a total of eight unique jean configurations.
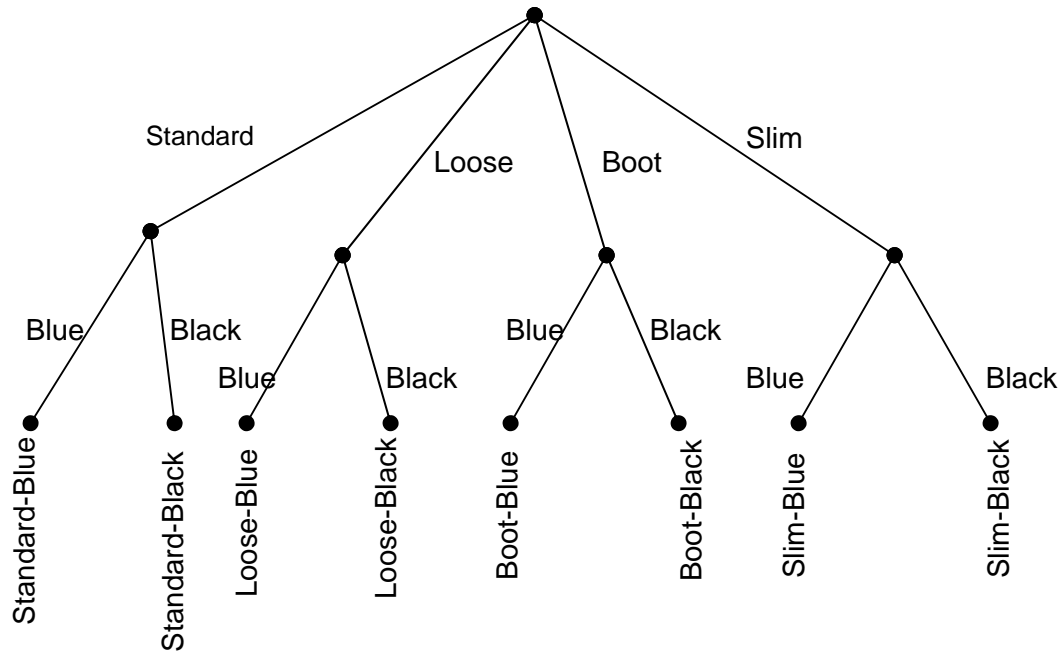

Figure 1-1: Enumeration of Jean Configurations using a Tree

A tree is one natural way to express possible configurations or outcomes when there are multiple choices that are possible. Alternately, the same information can be expressed in terms of a matrix. However, this format is only convenient if the number of features or choices is very limited (usually to two or three). Since there are only two choices in this case, it can easily be expressed in a table. Table 1-1 expresses the same information as Figure 1-1.

	Jean Style			
Color	**Standard**	**Loose**	**Boot**	**Slim**
Blue	Standard-Blue	Loose-Blue	Boot-Blue	Slim-Blue
Black	Standard-Black	Loose-Black	Boot-Black	Slim-Black

Table 1-1: Enumeration of Jean Configurations using a Table

The next section will provide rules to help with the counting process. While these rules are useful and are essential for solving complex problems, simple problems can often be solved by manually enumerating the possible outcomes.

Practice Example 1   *You are given three coins, a penny, a nickel and a dime. You toss all three high into the air and then record whether they land heads up or tails up. Enumerate the possible configurations of heads and tails.*

1.2 Elementary Rules for Counting

In this section we introduce two basic rules for counting: the multiplication rule and the addition rule. These rules are introduced through the use of examples. Before proceeding, however, we introduce some additional terminology. In the jean configuration example, the choices we had involved the style of the jean and the color of the jean. We referred to these as features or variables and each had a particular choice that could be chosen from a set of valid choices. We can also refer to these features or variables as *events* and the possible choices as *outcomes*. This terminology is more consistent with the terminology used in probability theory although it does not seem as natural for the jean example. In this section we adopt this more standard terminology.

1.2.1 The Multiplication Rule

We enumerated the number of configurations of jeans using a tree in Figure 1-1 and a table in Table 1-1. That example suggests a general rule for counting, the multiplication rule, which is defined below.

Multiplication Rule for Counting   If we have two events E_1 and E_2 and E_1 has n_1 possible outcomes and E_2 has n_2 possible outcomes, then we can enumerate are a total of $n_1 \cdot n_2$ possible outcomes for the two events. In general if we have v events $E_1 \ldots E_v$ with $n_1 \ldots n_v$ possible outcomes, then there will be a total of $n_1 \cdot n_2 \cdot \ldots \cdot n_v$ outcomes.

The multiplication rule for counting is only applicable if the events *all* must occur. More specifically, if there are v events, E_1 must occur and E_2 must occur, … and E_v must occur. This rule is applicable to the jean example described earlier, since one must specify the style of jean *and* the jean color. In English, the "clue" that this rule is applicable has to do with the use of the word "and". Thus one might want to think of this as the "and" rule for counting.

Now let us apply this to the jean example discussed earlier. In mathematics it is important to properly describe the problem and use the correct terminology. If this is done, coming up with the actual solution is sometimes trivial, as in this case.

Problem Statement:

Let E_1 = Choosing the jean style *and* E_2 = Choosing the jean color.

The outcomes for E_1 are {standard fit, loose fit, boot fit, and slim fit} and the outcomes for E_2 are {black, blue}.

Question: How many outcomes are there?

<u>Answer</u>: For this problem $n_1 = 4$ and $n_2 = 2$ so the number of outcomes is $4 \cdot 2 = 8$.

Note that in this case we were not asked to enumerate the outcomes, only list the total number of outcomes. In combinatorics we typically are only concerned with counting the number of outcomes, although you should always be able to enumerate them. You should be able to see the connection between the multiplication rule for counting and the tree structure in Figure 1-1: the number of leaves in the tree is $n_1 \cdot n_2$.

You should also be able to relate the enumeration process to your knowledge of set theory. In this case the enumeration of the outcomes could be represented by the cross product of the outcomes associated with sets E_1 and E_2. The number of items in the cross product of a set A and B is $|A| \cdot |B|$, so this agrees with the multiplication rule.

The best way to learn about combinatorics is with examples, so we include several additional examples in this section. When trying to solve these problems, remember that coming up with the problem statement is as important as the actual solution.

Example 1 *You are asked to flip a coin two times and to record the outcome (head or tail) for each flip. How many possible outcomes are there? Then enumerate the possible outcomes.*

There are two events, E_1 and E_2, corresponding to the two coin flips. Event 1 *and* event 2 must occur, so the multiplication rule is applicable. Each event has two possible outcomes, thus $n_1 = 2$ and $n_2 = 2$. Thus by the multiplication principle of counting, there are $2 \cdot 2 = 4$ possible outcomes.

The possible outcomes can be represented using pairs, such that (H,T) represents a head on the first flip and a tail on the second flip. The possible outcomes are then {(H,H), (H,T), (T, H), (T,T)}. Again, note that you could generate this by calculating the cross product of the sets of outcomes.

Example 2 *You are asked to flip a coin five times and to record the outcome (head or tail) for each flip. How many possible outcomes are there?*

This example differs from the previous one only in that there are five events instead of two. For each events there are two possible outcomes, so the total number of outcomes is $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^5 = 32$.

Example 3 *You play a lottery where you choose five numbers and each number must be between 1 and 20, inclusive. How many ways can you fill out the lottery card?*

In this example there are five events, corresponding to the five numbers that you must choose. Each of the five events must occur, so the multiplication rule is applicable. Each event has twenty possible outcomes (i.e., you pick a number between 1 and 20). Therefore, there are $20 \cdot 20 \cdot 20 \cdot 20 \cdot 20 = 20^5 = 3,200,000$ possible ways to fill out the lottery card.

Example 4   *You play a lottery where you choose five numbers and each number must be between 1 and 20, inclusive. However, once you choose a number, it is discarded and cannot be chosen again. How many ways can you fill out the lottery card?*

This example is identical to the previous one except that a number cannot be chosen more than once. This means that the number of possible outcomes for each event is reduced by one following each event. Assuming the five events E_1 … E_5 are numbered such that E_1 corresponds to the first number selected and E_5 to the last number selected, then the number of outcomes for E_1 is 20, for E_2 is 19, for E_3 is 18, for E_4 is 17 and for E_5 is 16. Thus the number of possible outcomes is $20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 = 1,860,480$. Note that this answer is a little more than half of the answer for the previous example.

The key point from this example is that the number of outcomes may change over time and may be affected by previous events. This will be a very common occurrence. Often, once an object is selected, it may not be re-selected.

1.2.2 The Addition Rule

We do not yet have an example of the addition rule for counting, so we introduce a new example. The example below is incredibly simple—so simple you may think it odd that it even warrants a rule.

Example 5   *You need to purchase one shirt of any kind. The store has five short sleeve shirts and eight long sleeve shirts. How many possible ways are there to choose a shirt?*

Since you can purchase a shirt of any kind, you can purchase either a short sleeve shirt *or* a long sleeve shirt. Since there are five ways to pick a short sleeve shirt and eight ways to pick a long sleeve shirt, there are $5 + 8 = 13$ outcomes (i.e., ways to pick a shirt).

Most people can solve this problem without any help. It embodies the addition rule, which you may want to think of as the "or" rule. This rule is formalized below.

Addition Rule for Counting   If we have two events E_1 and E_2 and E_1 has n_1 possible outcomes and E_2 has n_2 possible outcomes, then the total number of outcomes for E_1 or E_2 occurring is $n_1 + n_2$. In general, ff we have *v* events E_1 … E_v with n_1 … n_v possible outcomes, then the total number of ways of E_1 occurring or E_2 occurring … or E_v occurring is $n_1 + n_2 + … + n_v$.

While the multiplication rule can be thought of as the "and" rule, where all events must occur (E_1 *and* E_2 and E_3 …), the addition rule can be thought of as the "or" rule, where E1 occurs or E2 occurs or E3 occurs. Since this rule is straightforward, we provide no additional explanation. Nonetheless, we provide one more example below. Before

looking at the solution, consider whether the multiplication rule or addition rule is applicable.

Example 6 *How many ways are there to choose one additional class for your schedule if there are five evening classes <u>and</u> four days classes?*

In this example the event is choosing one class. You can choose a day class *or* an evening class, so the addition rule applies and there are $5 + 4 = 9$ ways to choose this class. Note that you could be misled by the use of "and" in the question (which was underlined to add to the confusion). The point of this example is that you must pay attention to what is really being asked and not only look at the terms that occur in the sentence. The key part is how the *events* are combined and in this case it is understood by the context of the question that you can pick a day *or* an evening class. The "and" in this case does not refer to relationship between the events.

1.2.3 Using the Elementary Rules for Counting Together

In this section we present additional counting problems that may be solved using a combination of the addition and multiplication rules. However, in some cases the problem can be solved using a single rule by viewing it in a certain way. This will be made clear by some of the examples.

Example 7 *How many odd three digit numbers are there, assuming that leading 0's are permitted (e.g., the number 008 is valid)?*

There are two nearly identical ways to solve this problem. These methods all have three events, related to choosing the first, second, and third digits, where the first digit is the most significant digit (i.e., in the hundreds column). All approaches make use of the fact that a number is odd if it ends with a 1, 3, 5, 7, or 9.

The most natural method only makes use of the multiplication rule. We simply choose values for E_1 and E_2 and E_3. How many choices are there? For E_1 and E_2 there are no restrictions so there are 10 choices for each, while there are only five choices for the E_3, which represents the "ones" digit. This yields $10 \cdot 10 \cdot 5 = 500$ outcomes.

However, we can view this problem in a slightly different manner. We say the number of outcomes is the number of outcomes where the number ends in a 1 *or* 3 *or* 5 *or* 7 *or* 9. That is, we break down the problem into five cases and then determine the number of outcomes for each of these five cases and then add them. In this case there are 100 outcomes from each $(10 \cdot 10)$ so we get 500.

The differences in the two approaches taken in Example 7 are very subtle, which can lead to some confusion. The next example may make things clearer since in this case it will be natural to apply both counting rules. The next question assumes you know the basics about a deck of cards and about poker. Since card and poker examples abound in combinatorics and probability, we provide a basic lesson on the subject here.

Brief lesson on cards and card games

A deck of cards contains 52 cards. Each card belongs to one of four suits (clubs, diamonds, hearts and spades) and is associated with one of thirteen denominations {2, 3, 4, 5, 6, 7, 8, 9, 10, J(ack), Q(ueen), K(ing), A(ce)}. The clubs and spades are black and the diamonds and hearts are red. Unless otherwise specified, assume that for any example you begin with a complete deck and that as cards are dealt, they are not immediately replaced back into the deck. We abbreviate a card using the denomination and then suit, such that 2H represents a 2 of Hearts.

In standard poker you receive 5 cards. While you can later discard cards and then replace them, for most of our examples we will only consider the initial configuration. A "pair" in poker means that you have two cards that are the same denomination, such as a pair of 4's. Three-of-a-kind and four-of-a-kind are defined similarly, based on the denomination. A flush means that all of the cards are of the same suit (usually 5 cards in poker).

Example 8 *How many ways can you draw a flush in poker assuming that the order of the five cards drawn matters (we will learn how to relax this assumption in the next section).*

Since a flush means that all five cards are of the same suit and there are four suits, there are four basic ways to get a flush: all clubs or all diamonds or all hearts or all spades. We can view each of these as events and our goal is to determine the total number of outcomes of these four non-overlapping events. The number of ways we can get five of the same suit is the same for all suits, so we only need to compute the number for any suit and then multiply the answer by the number of suits, which is four. Below we arbitrarily select clubs and figure out the number of ways to draw five clubs. Note that we start with 13 clubs in the deck. Using the multiplication rule to select 5 cards, below we compute the number of ways to draw five clubs:

Number of ways to draw five clubs = $13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 = 594$.

Therefore, by the addition rule, the number of ways to get a flush is $4 \cdot 594 = 2,376$.

1.3 Permutations and Combinations

When counting, we sometimes care about the order of events while sometimes we do not. This makes a big difference when counting. To see this, imagine that after selecting five cards one has the following cards in one's hand: 2H, 3S, 4D, 5C, 6H. If we do not care about order, then when counting, we count this configuration once, regardless of what order the cards were drawn in. However, if we care about the sequence in which the cards were drawn, then there are many sequences that can be formed containing only these cards and all of these should be considered to be distinct. How many ways could we draw the five cards listed above from a deck of cards? Using the multiplication rule for counting, there are 5 possibilities for the first draw, 4 for the second, 3 for the third, two

for the fourth and one for the last. This yields $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ ways. Thus we see that if order does matter, we will always get a bigger number than if order does not matter. In this particular example, the answer we get is 1 if order does not matter and 120 if order does matter. For every problem in combinatorics that you see, you should ask yourself whether order matters or does not matter. Sometimes the example will mention this explicitly, whereas sometimes you must determine this from the context of the problem. In all of the examples so far, we have assumed that order *does* matter. If order does not matter, then the answer we come up with must be reduced. We will shortly see how to do this. In combinatorics, order is important for permutations but not for combinations.

1.3.1 Permutations

A permutation of objects is an arrangement where order, or position, does matter. For example, the telephone number 123-456-7890 is not the same as the telephone number 789-012-3456 so it makes sense to talk about permutations when discussing phone numbers. In a permutation of objects order matters and each object can only be used once. Permutation problems are just one specific kind of counting problem. As an example, imagine that we have five children and five chairs in a row. Instead of asking how many ways we can place the children into the five chairs, we could simply ask how many permutations of the five children there are. Not that this problem has the meets the key criteria for a permutation. First, order does matter (in this case it can be inferred from the problem). Second, a child cannot sit in two chairs at once. Using our knowledge of counting, we know by the multiplication rule that there are $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ permutations. Now what if we have 10 children and want to place them into the five chairs? In this case the answer would be $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 = 30{,}240$. There is a general formula to calculate permutations—although we could always compute it from scratch using the multiplication rule.

The number of permutations for *r* items chosen from a total of *n* items is denoted P(*n*,*r*). The number of permutations for 10 children and 5 chairs is P(10,5). We could also say this slightly differently—the number of ways to permute 10 children into 5 chairs is P(10,5). The number of ways to permute 5 children into 5 chairs is denoted P(5,5).

The formula for calculating P(*n*,*r*) is given below:

$$P(n,r) = \frac{n!}{(n-r)!}$$

In this equation the "!" refers to a factorial and n! refers to "n factorial". The value of a number x! is defined as x(x-1)(x-2) …. (1) and 0! is defined to be 1. Below are some examples:

$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$
$10! = \ 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 3{,}268{,}800$

Using the formula for a permutation, we see that the number of permutations of 10 children in 5 seats, P(10,5) = 10!/(10-5)! = 10!/5!. Rather than computing 10! and 5! and then dividing the two, we can save some effort by crossing off the common elements. Thus we get:

$$P(10,5) = \frac{10!}{(10-5)!} = \frac{10!}{5!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot \cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}}{\cancel{5} \cdot \cancel{4} \cdot \cancel{3} \cdot \cancel{2} \cdot \cancel{1}} = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 = 30{,}240$$

Note that this is the same answer we computed using the multiplication rule. It is important that you be able to compute these answers from "first principles", but once you have seen enough examples it is acceptable to just use the formula for permutations.

Example 9 *If a baseball team has 25 players, how many batting orders are there?*

First, for those not familiar with baseball, a batting order consists of 9 players and order does matter (i.e., it matters who bats first, second, etc.). Since a batter cannot bat twice in a lineup, the question asks how many permutations there are for 25 players into 9 spots (the rest of the players are on "the bench"). The answer is P(25,9) = 25!/16! = 25 · 24 · … · 17 = 741,354,768,000.

1.3.2 Combinations

Permutations assume that order matters. For a baseball lineup order does matter, but order does not matter if instead one is selecting members to be on a jury or on a committee—assuming that all members are equal and have identical roles. When order does not matter, we care about combinations, not permutations. If we need to choose *r* items from *n* and order does not matter, we denote this C(*n*,*r*). Thus, if we have a jury pool of 100 people and need to choose 12, the number of combinations would be C(100,12), read as "100 choose 12." The formula for computing combinations is shown below. Note that, as with permutations, an item cannot be selected more than once.

$$C(n, r) = \frac{P(n,r)}{r!} = \frac{n!}{r!(n-r)!}$$

The only difference between the formula for permutations and combinations is that the formula for combinations has an additional r! in the denominator. Given that we previously concluded that the number of outcomes should be reduced if order does not matter, this makes sense. But why is it reduced by r!? To see this, we will continue the baseball example.

If we want to compute the number of ways to choose 9 people from 25 where order does not matter, we get C(25,9) = 25!/(9!·16!). Thus, the number is reduced by a factor of 9! when compared to the number of permutation. The reason is quite simple: 9! of the individual outcomes when order does matter become equivalent when order does not matter. To see this, imagine you have already selected 9 people. How many ways can you order them? You can order them 9 · 8 · 7 · 6 · 5 · 4 · 3 · 2 · 1 = 9! ways. In general, r items can be ordered in r! ways, so that is why you must divide by r! when order does not matter.

Example 10 *How many ways can a teacher choose 4 students out of the 20 in her class?*

Since there is no indication that order matters, we assume that order does not matter. Therefore the answer is C(20,4) = 20!/(4!·16!) = (20·19·18·17)/4! = (20·19·18·17)/4! = 5·19·3·17 = 4,845 (in the last step we divide 20 by 4 and 18 by 3·2 to get rid of the denominator).

Example 11 *How many ways can a university choose a freshman class president and vice-president if there are 250 freshman students and all are eligible?*

The first question we need to ask here is if order matters and whether we want to compute combinations or permutations. The answer here is that effectively order does matter since there are two distinct roles to be filled. That is, there is a big difference between the case that John Smith is president and Mary Smith is vice-president and the case that Mary Smith is president and John Smith is vice-president. Since these two cases are different, we must effectively consider the order. Since there are 250 choices for president and then 249 choices for vice-president, the answer is 250·249, which is the same as P(250,2). Thus, when determining whether to use permutations or combinations, use permutations if the order matters or the assigned roles to be filled are distinguishable.

1.4 Additional Examples

The examples up until now have been relatively straightforward. In this section we include some more difficult examples and do not specify ahead of time whether you should use the multiplication or addition principle, or combinations or permutations.

Example 12 *A typical telephone number has 10 digits (e.g., 555-817-4495), where the first three are known as the area code and the next three as the exchange. Answer the following questions:*

   a. *Assuming <u>no</u> restrictions, how many possible (3 digit) area codes are there?*

   b. *Assuming that the middle digit of the area code must be a 0 or a 1 (which was required until recently), how many possible (3 digits) area codes are there?*

   c. *Assuming no restrictions whatsoever, how many possible values are there for the full 10-digit phone number?*

   d. *If the only restriction is that no digit may be used more than once, how many possible 10-digit phone numbers are there?*

   Solution:

   a. With no restrictions there are $10 \cdot 10 \cdot 10 = 1,000$ 3-digit area codes

   b. If middle digit must be 0 or 1, there are $10 \cdot 2 \cdot 10 = 200$ area codes

   c. With no restrictions there are $10^{10} = 10,000,000,000$ phone numbers

   d. If each digit can only be used once there are $10 \cdot 9 \cdot \ldots \cdot 1 = 10!$ Numbers

Example 13 *Answer the following given that a class has 15 Women and 10 Men.*

    a. How many ways are there to choose one class member to take attendance?

    b. How many ways are there to choose 2 people to clean the board?

    c. How many ways are there to choose one person to take attendance and one person to clean the board?

    d. How many ways are there to choose one person to take attendance and one person to clean the board if both jobs <u>cannot</u> be filled with people of the same sex?

    e. How many ways are there to choose a person to erase the board and a person to take attendance if both of the jobs <u>must</u> be filled with people of the same sex?

Solution:

    a. $15 + 10 = 25$ (addition rule)

    b. $C(25,2) = 25! / (2! \cdot 23!) = (25 \cdot 24)/2! = 25 \cdot 12 = 300$ (order does not count)

    c. $P(25,2) = 25!/23! = 25 \cdot 24 = 600$ (different roles so order effectively counts)

    d. $15 \cdot 10 + 10 \cdot 15 = 300$ (additional restriction requires you do it from scratch)

    e. $15 \cdot 14 + 10 \cdot 9 = 300$ (addition rule on male and female case)

Example 14 *How many distinguishable ways are there to arrange the letters in the word "Mississippi"?*

This question is different from the other permutation and combination questions we have seen thus far. Based in the multiplication rule, we can arrange the letters in 11! ways. However, not all of these ways are distinguishable since some letters repeat ("s" and "i" each occur 4 times and p occurs 2 times) and when a letter repeats, we are overcounting the number of distinguishable configurations if we treat these letters as if they were all different. For example, let's say we place all of the 4 s's at the end of the word. With the naïve method we are counting this one distinguishable configuration $4 \cdot 3 \cdot 2 \cdot 1 = 24$ times instead of only one time, since there are 4! ways to permute these 4 s's. There are two approaches that we can take to answer this question. We can 1) calculate things naively and then correct for the fact that some of the permutations are indistinguishable or 2) calculate things to avoid the problem in the first place. We employ both approaches but recommend that in the future, for similar problems you use the second approach.

<u>Approach 1</u>: The naïve answer is 11!. We then account for the fact that we have 4 s's by dividing this answer by 4! and then again divide by 4! due to the 4 i's. We also divide by 2! to account for the 2 p's. Thus, our final answer is $11!/(4!4!2!) = 34{,}650$.

Approach 2: In this approach we view this as a fill-in-the-blank problem with 11 blanks and then carefully see how many ways we can fill in the blanks. We view this as a combination problem, but here we choose the blanks to fill in, not the letters to use. The best way to follow this is to work through the example. We start with the 11 blanks:

___ ___ ___ ___ ___ ___ ___ ___ ___ ___ ___

We then start assign letters to the blanks. We will pick the easiest letters first (i.e., the ones with the fewest occurrences). How many ways can we assign the 1 M to the blanks above? The answer is $C(11,1) = 11!/10!1! = 11$. As we expect, there are 11 ways. Once this is done, we next decide how many distinct ways to assign the 2 p's. Since there are only 10 slots left, the answer is $C(10,2)$. Note that because the 2 p's are indistinguishable, we use $C(10,2)$ instead of $P(10,2)$. We then assign the 4 s's to the 8 remaining spots in $C(8,4)$ ways. Then we assign the 4 i's to the 4 remaining spots in $C(4,4)$ ways (note there is only one way to do this since there are 4 i's and only 4 spots left). The total number of ways is then: $C(11,1) \cdot C(10,2) \cdot C(8,4) \cdot C(4,4) = 11 \cdot 45 \cdot 70 \cdot 1 = 34{,}650$.

Exercises for Chapter 1

1. A Fordham University club has 25 members of which 5 are freshman, 5 are sophomores, 10 are juniors and 5 are seniors.

   a. How many ways can a president be selected if freshman are ineligible to be president?

   b. How many ways are there to select two seniors to serve on the College Council?

   c. How many ways are there to choose a president and vice-president if the president and vice-president cannot be from the same class and freshman cannot hold either position?

   d. How many ways are there to choose a subcommittee with 25 members?

2. A new car can be ordered with a choice of 10 exterior colors, 7 interior colors, with or without air-conditioning, with or without power steering, and with or without a power package. Assuming all possible combinations are allowable, how many possible versions of this car can one order?

3. You toss two standard six-sided dice, one red and one blue. How many outcomes are there where the red die has a higher value then the blue die?

4. You toss two standard six-sided dice, both identical in every way. How many ways can you toss a seven (i.e., the sum of both rolls is a 7)? How many ways can you toss a 2 (i.e., a snake eyes)?

5. Your friend has 15 magazines and says you can borrow any 5 magazines. How many different possible collections of magazines could you choose?

6. Forty slips of paper numbered 1 through 40 are placed in a hat and drawn out. How many different pairs of numbers can be drawn?

7. If the automobile license plate must be of the form XXX-DDDD where X is a capital letter and D is a digit (0-9), then how many unique license plates are there?

8. Suppose there are ten applicants, seven men and three women, for a job. The men and women will be interviewed in a random order. How many different ways can they be interviewed?

9. How many distinguishable arrangements of the word "radar" are there?

10. How many 10 digits binary numbers have exactly 3 1's?

11. ** How many ways are there to seat 10 people at a round table, if we don't care about the absolute position of anyone, only their position relative to the others (i.e., having everyone shift one position to the right does not yield a different configuration)?

12. *** A bag has 32 balls – 8 each or orange, white, red and yellow. All balls of the same color are indistinguishable. A juggler randomly picks three balls from the bag to juggle. How many possible groupings of balls are there?