CISC 4090
Theory of Computation

Finite state machines & Regular languages

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JMH 332

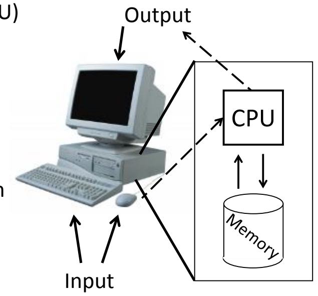
Stereotypical computer

Central processing unit (CPU) – performs all the instructions

Memory – stores data and instructions for CPU

Input – collects information from the world

Output – provides information to the world



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Super-simple computers

Small number of potential inputs
Small number of potential outputs/actions

- Thermostat
- Elevator
- Vending machine
- Automatic door

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Automatic door

Desired behavior
- Person approaches entryway, door opens
- Person goes through entryway, door stays open
- Person is no longer near entryway, door closes
- Nobody near entryway, door stays closed

Two states: Open, Closed
Two inputs: Front-sensor, Back-sensor

Finite state machine

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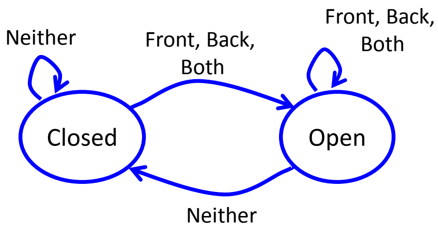
Graph and table representations



	Front	Back	Neither	Both
Closed	Open	Open	Closed	Open
Open	Open	Open	Closed	Open

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More finite state machine applications

- Text parsing
- Traffic light
- Pac-Man
- Electronic locks

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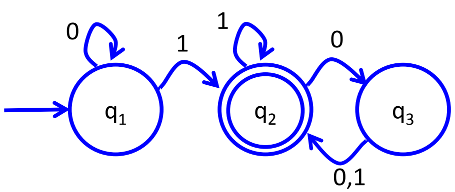
Coding a combination lock



- A finite automaton M1 with 3 states
- Start state q1; accept state q2 (double circle)
- Example accepted string: 1101
- What are all strings that this model will accept?

String ending with 1 or string containing 1 and ending with 00

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Formal definition of Finite State Automaton

Finite state automaton is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$

- Q is a finite set called states
- Σ is a finite set called the alphabet
- $\delta: Q \times \Sigma \longrightarrow Q$ is the transition function
- $q_0 \in Q$ is the start state
- $F \subseteq Q$ is the set of accept states

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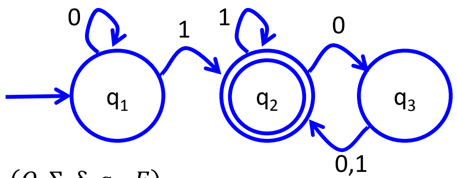
Describe M1 using formal definition



$M1 = (Q, \Sigma, \delta, q_0, F)$

- $Q = \{q_1, q_2, q_3\}$
- $\Sigma = \{0, 1\}$
- Start state: q_1
- $F = \{q_2\}$
- $\delta =$

	0	1
q_1	q_1	q_2
q_2	q_3	q_2
q_3	q_2	q_2

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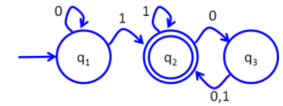
Language of M1



If A is set of all strings accepted by M, A is language of M
- L(M)=A

A machine may accept many strings, but only one language
- M **accepts** a string
- M **recognizes** a language

Describe L(M1)=A
- A={w|**w ends with 1 or w contains at least one 1 and ends in 00}**

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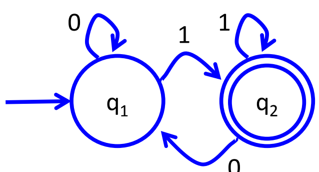
Describe M2 using formal definition



$M1 = (Q, \{0,1\}, \delta, q_0, \{q_2\})$

- $Q = \{q_1, q_2\}$
- Start state: q_1
- $\delta =$

	0	1
q1	q1	q2
q2	q1	q2

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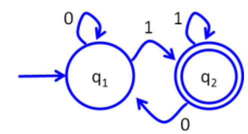
What is the language of M2?



L(M2)={w| w ends with at least one 1}

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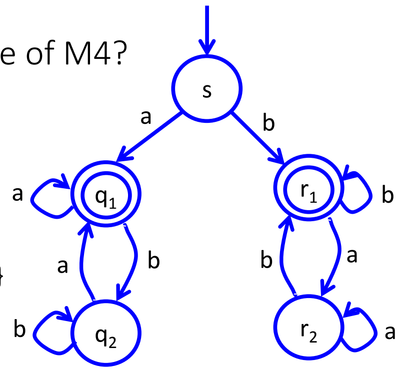
What is the language of M4?

(page 38, Ex. 1.11)

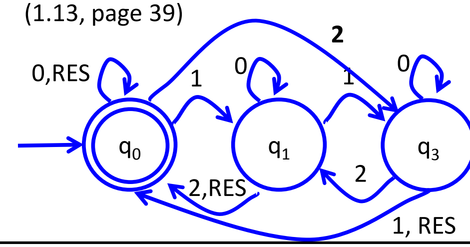
L(M4)={w| w ends and begins with same letter (either a or b)}



Perform modulo arithmetic

Let Σ={RESET, 0, 1, 2}

Construct M5 to accept a string only if the sum of each input symbol is multiple of 3, and RESET sets the sum back to 0 (1.13, page 39)



More modulo arithmetic

Generalize M5 to accept if sum of symbols is a multiple of *i* instead of 3

$(\{q0, q1, q2, q3, \cdots, q_{i-1}\}, \{0,1,2,\text{RESET}\}, \delta, q0, F)$

$\delta(qj, RESET) = q0$

$\delta(qj, 0) = qj$

$\delta(qj, 1) = qk$ for k = j+1 mod i

$\delta(qj, 2) = qk$ for k = j+2 mod i

Definition of M accepting a string

Let M $= (Q, \Sigma, \delta, q0, F)$ be a finite automaton and let $w = w_1 w_2 \cdots w_n$

Then M accepts w if a sequence of states $r_0, r_1, \ldots, r_n$ in Q exists with 3 conditions

- $r_0 = q_0$
- $\delta(r_i, w_{i+1}) = r_{i+1}$ for $i = 0, 1, \cdots, n-1$
- $r_n \in F$

Regular languages

Definition: a language is called a regular language if some finite automaton recognizes it

equivalently

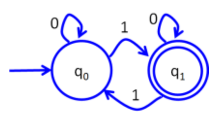
All of the strings in a regular language are accepted by some finite automaton

Designing finite automata (FAs)

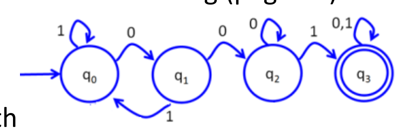
- Determine what you need to remember
  - How many states needed for your task?
- Set start and finish states
- Assign transitions
- Check your solution
  - Should accept $w \in L$
  - Should reject $w \notin L$
  - Be careful about ε!

FA design practice!

- FA to accept language where number of 1's is odd (page 43)

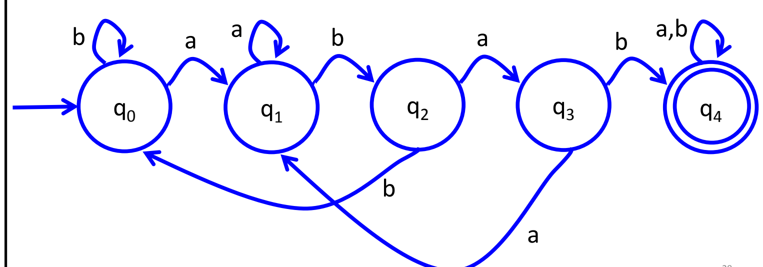


- FA to accept string with 001 as substring (page 44)



- FA to accept string with substring abab **(next page!)**

FA to accept string with substring abab



Regular operations

Let A and B be languages. We define 3 regular operations:

- Union: $A \cup B = \{x | x \in A \text{ or } x \in B\}$
- Concatenation: $A \cdot B = \{xy | x \in A \text{ and } y \in B\}$
- Star: $A^* = \{x_1 x_2 \cdots x_k | k \geq 0 \text{ and each } x_i \in A\}$
  - Repeat a string 0 or more times

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Examples of regular operations

Let $A = \{\text{good}, \text{bad}\}$ and $B = \{\text{boy}, \text{girl}\}$

What is:

- $A \cup B = \{\text{good}, \text{bad}, \text{boy}, \text{girl}\}$
- $A \cdot B = \{\text{goodboy}, \text{goodgirl}, \text{badboy}, \text{badgirl}\}$
- $A^* = \{\varepsilon, \text{good}, \text{bad}, \text{goodgood}, \text{goodbad}, \text{badgood}, \text{badbad}, \cdots\}$

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Closure

A collection of objects is closed under an operation if applying that operation to members of the collection returns an object in the collection

Regular languages are closed under $\cup$, $\cdot$, $*$

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Closure of Union

Theorem 1.25: The class of regular languages is closed under the union operation

Proof by construction

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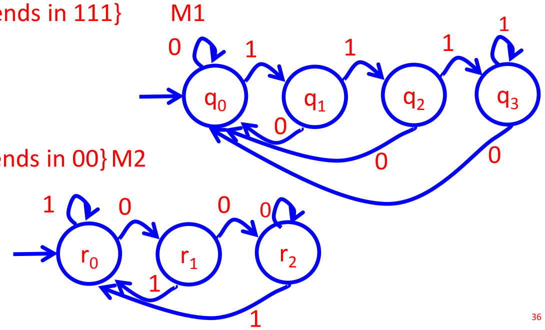
Example union

A = {w | w ends in 111} M1

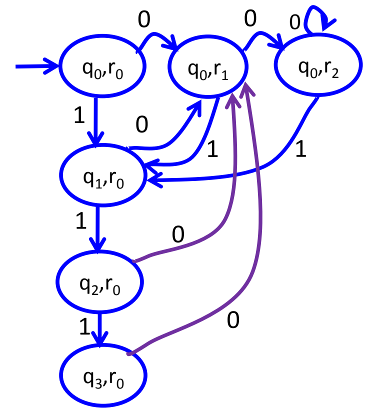
B = {w | w ends in 00} M2



Example union

AUB M5

Simulate M1 and M2 states



Closure of Union – Proof by Construction

Let us assume M1 recognizes language L1

- Define M1 as $M1 = (Q, \Sigma, \delta_1, q_0, F_1)$

Let us assume M2 recognizes language L2

- Define M2 as $M2 = (R, \Sigma, \delta_2, r_0, F_2)$

Proof by construction: Construct M3 to recognize $L3 = L1 \cup L2$

- Let M3 be defined as $M3 = (S, \Sigma, \delta_3, s_0, F_3)$

Closure of Union – Proof by Construction

- Let M3 be defined as $M3 = (S, \Sigma, \delta_3, s_0, F_3)$

Use each state of M3 to simulate being in a state of M1 and another state in M2 simultaneously

M3 states: $S = \{(q_i, r_j) \mid q_i \in Q \text{ and } r_j \in R\}$

Start state: $s_0 = (q_0, r_0)$

Accept state: $F_3 = \{(q_i, r_j) \mid q_i \in F_1 \text{ or } r_j \in F_2\}$

Transition function: $\delta_3\left((q_i, r_j), x\right) = \left(\delta_3(q_i, x), \delta_3(q_j, x)\right)$

Closure of Concatenation

Theorem 1.26: The class of regular languages is closed under the concatenation operation

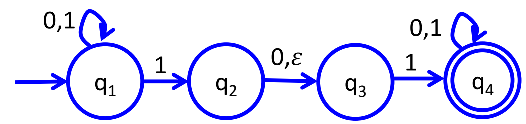
- If A1 and A2 are regular languages, then so is $A1 \cdot A2$
- Challenge: How do we know when M1 ends and M2 begins?

Determinism vs. non-determinism

Determinism: Single transition allowed given current state and given input

Non-determinism:

- multiple transitions allowed for current state and given input
- transition permitted for null input ε



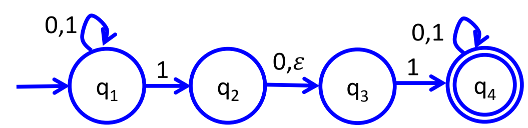
NFA in action

- When there is a choice, follow all paths – like cloning
- If there is no forward arrow, path terminates and clone dies (no accept)
- NFA will “accept” if at least one path terminates at accept

Alternative thought:

- Magically pick best path from the set of options

The language of M10



- List some accepted strings

  110 – at third entry, we’re in states {q_1,q_3, and q_4}
- What is L(M10)?

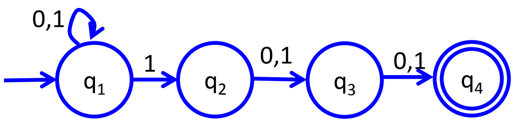
  {w | w contains 11 or 101}

NFA construction practice

Build an NFA that accepts all strings over {0,1} with 1 in the third position from the end

NFA construction practice

Build an NFA that accepts all strings over {0,1} with 1 in the third position from the end



If path is at q_4 and you receive more input, your path terminates

NFA -> DFA

Build an NFA that accepts all strings over {0,1} with 1 in the third position from the end

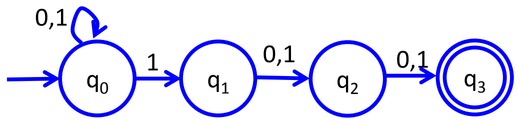
Can we construct a DFA for this?

Formal definition of Nondeterministic Finite Automaton

Similar to DFA: a 5-tuple $(Q, \Sigma, \delta, q_0, F)$

- Q is a finite set called states
- Σ is a finite set called the alphabet
- $\delta: Q \times \Sigma\varepsilon \longrightarrow P(Q)$ is the transition function
- $q_0 \in Q$ is the start state
- $F \subseteq Q$ is the set of accept states

Describe M10 using formal definition



$M1 = (Q, \Sigma, \delta, q_0, F)$

- $Q = \{\mathbf{q_0}, \mathbf{q_1}, \mathbf{q_2}, \mathbf{q_3}\}$
- $\Sigma = \{\mathbf{0}, \mathbf{1}\}$
- Start state: $\mathbf{q_0}$
- $F = \{\mathbf{q_3}\}$
- $\delta =$

	0	1	ε
q_0	$\{q_0\}$	$\{q_0,q_1\}$	{}
q_1	$\{q_2\}$	$\{q_2\}$	{}
q_2	$\{q_3\}$	$\{q_3\}$	{}
q_3	{}	{}	{}

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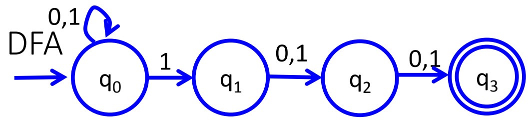
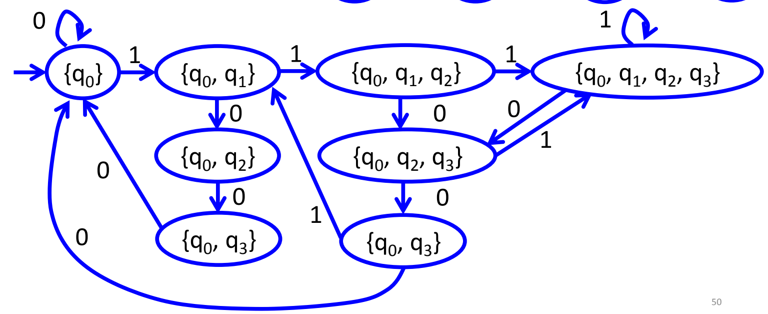
Describe M10 as DFA





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Equivalence of NFAs and DFAs

NFAs and DFAs recognize the same class of languages

Two machines are equivalent if they recognize the same language

Every NFA has an equivalent DFA

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Equivalence of NFAs and DFAs

NFA $\quad N1 = (Q, \Sigma, \delta, q_0, F)$

Define DFA $\quad M1 = (R, \Sigma, \delta^D, r_0, F^D)$

- **R=P(Q)** --- $R = \{\{\}, \{q_0\}, \ldots, \{q_n\}, \{q_1,q_2\}, \ldots \{q_{n-1},q_n\}, \ldots\}$
  every combination of states in Q
- $r_0=\{q_0\}$
- $F^D = \{s \in R \mid s \text{ contains at least 1 accept state for N1}\}$
- $\delta^D(r_i, x)$ **Consider all states q_j in r_i, find r_k that is union of outputs for N1's $\delta(q_j, x)$ for all q_j**

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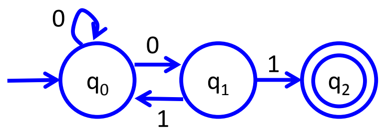
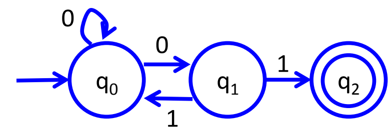
Consider NFA N1

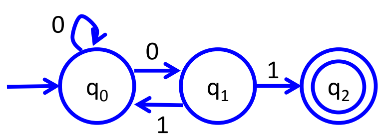


Language:
L(N1)={w | w begins with 0, ends with 01, every 1 in w is preceded by a 0}



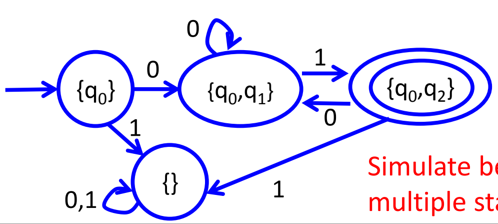
Convert NFA N1 to DFA M1

N1



Convert NFA N1 to DFA M1

N1



M1

Simulate being in multiple states at once

Union Closure with NFAs

- Proofs by construction – fewer states!
- Any NFA proof applies to DFA

Given two regular languages A_1 and A_2 recognized by N1 and N2 respectively, construct N to recognize $A_1 \cup A_2$

Let's consider two languages

L1: start with 0, end with 1

L2: start with 1, end with 0

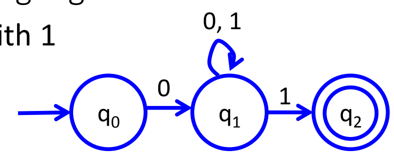
Construct machines for each languages

Construct machines N3 to recognize L1 U L2

Let's consider two languages

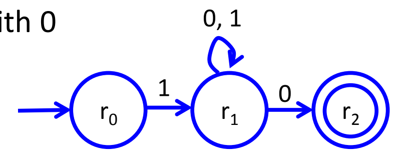
L1: start with 0, end with 1

N1

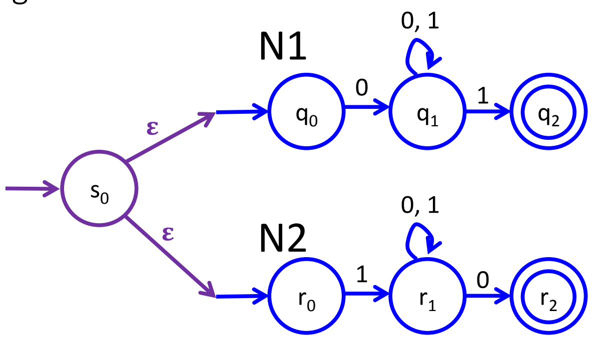


L2: start with 1, end with 0

N2



N3 recognizes L1 U L2



Closure of regular languages under union

This is a good example of how to write up a general proof by construction

Let N1 = $(Q, \Sigma, \delta_1, q_0, F_1)$ recognize L1

Let N2 = $(R, \Sigma, \delta_2, r_0, F_2)$ recognize L2

N3 = $(Q_3, \Sigma, \delta_3, s_0, F_3)$ will recognize L1 U L2 iff

$Q_3 = Q \cup R \cup \{s_0\}$

Start state: s_0

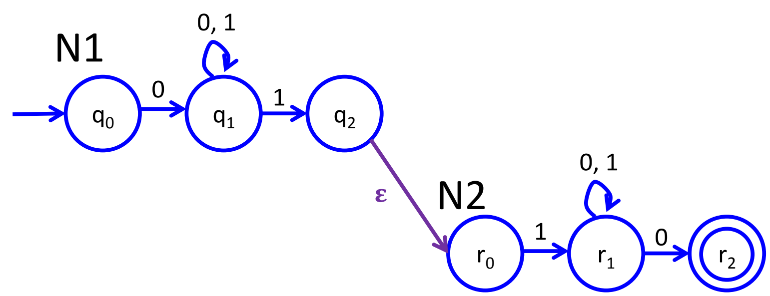
$F_1 = F_2 \cup F_3$

$$\delta_3(q, a) = \begin{cases} \delta_1(q, a) & \text{if } q \in Q \\ \delta_2(q, a) & \text{if } q \in R \\ \{q_0, r_0\} & \text{if } q = s_0 \text{ and } a = \varepsilon \end{cases}$$

Closure under concatenation

Given two regular languages A_1 and A_2 recognized by N1 and N2 respectively, construct N to recognize $A_1 \cdot A_2$

Concatenation: $L_1 \cdot L_2$



Closure of regular languages under concatenation

This is a good example of how to write up a general proof by construction

Let N1 = $(Q, \Sigma, \delta_1, q_0, F_1)$ recognize L1

Let N2 = $(R, \Sigma, \delta_2, r_0, F_2)$ recognize L2

N3 = $(Q_3, \Sigma, \delta_3, s_0, F_3)$ will recognize $L_1 \cdot L_2$ iff

$Q_3 = Q \cup R$

Start state: q_0

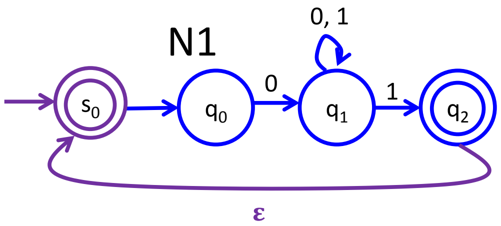
$F_1 = F_3$

$$\delta_3(q, a) = \begin{cases} \delta_1(q, a) & \text{if } q \in Q \\ \delta_2(q, a) & \text{if } q \in R \\ r_0 & \text{if } q \in F_1 \text{ and } a = \varepsilon \end{cases}$$

Closure under star

Prove if A_1 is regular, A_1^* is also regular

Star: L_1^*



Closure of regular languages under star

This is a good example of how to write up a general proof by construction

Let N1 = $(Q, \Sigma, \delta_1, q_0, F_1)$ recognize L1

N3 = $(Q_3, \Sigma, \delta_3, s_0, F_3)$ will recognize L1* iff

$Q_3 = Q \cup \{s_0\}$

Start state: s_0

$F_1 = F_3 \cup \{s_0\}$

$$\delta_3(q, a) = \begin{cases} \delta_1(q, a) & \text{if } q \in Q \\ q_0 & \text{if } q = s_0 \text{ and } a = \varepsilon \\ s_0 & \text{if } q \in F_1 \text{ and } a = \varepsilon \end{cases}$$

Regular expressions

A regular expression is description of a set of possible strings using a single characters and possibly including regular operations

Examples:

- $(0 \cup 1)0^*$
- $(0 \cup 1)^*$

Regular expressions – formal definition

R is a regular expression if R is

- a, for some a in alphabet Σ
- ε
- $\emptyset$
- $R1 \cup R2$, where R1 and R2 are regular expressions
- $R1 \cdot R2$, where R1 and R2 are regular expressions
- $R1^*$, where R1 is a regular expression

This is a recursive definition

Examples of Regular Expressions

- 0^*10^*={1, 010, 100, 00100, 001, …} =
  {w | w contains exactly one 1}
- $\Sigma^*1\Sigma^*$={1,11,01,011,001,110,111,…} =
  {w | w contains at least one 1}
- $01 \cup 10$={01, 10}
- $(0 \cup \varepsilon)(1 \cup \varepsilon)$={01,0,1,$\varepsilon$}

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FA can recognize any Regular Expression

Theorem: A language is regular if and only if some regular expression describes it

- Prove: If a language is described by a regular expression, then it is regular
- Prove: If a language is regular, then it is described by a regular expression

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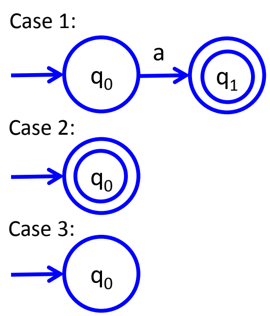
Prove if language described regular expression, it is regular (recognized by FSA)

Each regular expression is either

- Case 1: $a \epsilon \Sigma$
- Case 2: ε
- Case 3: ∅
- Case 4: R1 ∪ R2 – Theorem 1.45
- Case 5: R1 · R2 – Theorem 1.47
- Case 6: R1* – Proven on slide 50



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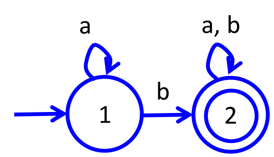
Converting from FSA to Regular Expression



$a^*b(a \cup b)^*$

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