CISC 4090
Theory of Computation

Non-regular languages

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JMH 332

Regular languages

Definition: a language is called a regular language if some finite automaton recognizes it

What languages cannot be recognized by an FSA

Regular languages use finite memory (finite states)
Non-regular languages require infinite memory

Are the following regular?

$L_1 = \{w \mid w \text{ has at least 100 1's}\}$
Yes: Start at $q_0$, For each 1 $q_k \rightarrow q_{k+1}$. $F = \{q_{100}\}$

$L_2 = \{w \mid w \text{ has same number of 0's and 1's}\}$
No: unknown number of states

$L_3 = \{w \mid w \text{ is of the form } 0^n1^n, n>0\}$
No: unknown number of states

What about this class of languages

$\Sigma = \{a, b\}$
$L_n = \{w \mid w \text{ contains } n \text{ b's in a row}\}$
• $L_1 = \{abbba, aabbbba, ababbbba, \ldots\}$
• $L_4 = \{babbbab, bbbb, aaabbbab, \ldots\}$

$L_n$ is regular for each value of $n$
Regular languages can be infinite
• E.g., a(ba)*b

For FSA to generate an infinite set of strings, there must be a loop between some states

Pumping lemma

Every string in regular language L with length greater than or equal to the pumping length p can be “pumped”

Every string s ∈ L (|s|≥p) can be written as xyz where
1. For each i ≥ 0, xy^iz ∈ L
2. |y| > 0
3. |xy| ≤ p

If L violates pumping lemma, then it is not regular

Proof idea

If |s| ≤ p, trivially true

If |s|>p, consider the states the FSA goes through
• Since there are only p states, |s|>p, one state must be repeated
  • Pigeonhole principle: There must be a cycle
Prove \( B = \{ 0^n1^n \} \) is not regular

Proof by contradiction: assume \( B \) is regular

thus, any \( w \in B \) can be “pumped” if \(|w| > p\)

First suggestion: \( w = 001 \), \( x = 0 \), \( y = 01 \), \( z = 1 \) – counterexample

\[ xy^2z = 001011 \notin B \]

Close! But maybe \(|001| \leq p\), how do we know this will be problem when \(|w| > p\)

Our solution: Let \( w = 0^p1^p |w| > p \), so must be “pump”-able

\[ |xy| \leq p \text{ so, } x = 0^f \text{ } y = 0^g, \text{ } f + g \leq p \text{ and } g > 0 \]

When we pump \( w \): \( xy^2z \), we get \( p + g \) 0’s followed by \( p \) 1’s. \( xy^2z \notin B \)

Contradiction, pumped \( w \notin B \)

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Common pumping proof-by-contradiction

Define a simple word \( w \) that is guaranteed to have more than \( p \) symbols, and you know the first \( p \) symbols

Show repetition of intermediate \( y \) string violates language rules

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Prove \( F = \{ \text{ww} \mid w = 0 \cup 1^* \} \) is not regular

Proof by contradiction: assume \( F \) is regular

thus, any \( v \in F \) can be “pumped” if \(|v| > p\)

• Our solution: Let \( w = 0^p1^p |w| > p \) so must be “pump”-able

\[ |xy| \leq p \text{ so, } x = 0^f \text{ } y = 0^g, \text{ } f + g \leq p \text{ and } g > 0 \]

When we pump \( w \): \( xy^2z \), we get \( p + g \) 0’s followed by \( 10^p1 \). \( xy^2z \notin B \)

Contradiction, pumped \( w \notin F \)

\( F = \{11, 00, 0101, 1010, 11011101, \ldots\} \)

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Prove \( E = \{ 1^{n^2} \} \) is not regular

Proof by contradiction: assume \( E \) is regular

thus, any \( w \in E \) can be “pumped” if \(|w| > p\)

Our solution: Let \( w = 1^p \), \( |w| > p \), so must be “pump”-able

\[ |xy| \leq p \text{ so } |y| \leq p \]

\[ |xy^2z| \leq p^2 + p \]

What’s the length of the next-biggest string after \(|w| = p^2 \)

\[ |w^{next\text{-}biggest} | = (p+1)^2 = p^2 + 2p + 1 \]

Pumping \( w \) once gives length at most \( p^2 + p < p^2 + 2p + 1 \)

Thus, \( xy^2z \notin E \)

Contradiction, pumped \( w \notin E \)
Prove $A=\{0^i1^j \mid i>j>0\}$ is not regular

Proof by contradiction: assume $A$ is regular
thus, any $w \in A$ can be "pumped" if $|w|>p$

Our solution:

Let $w=0^{p+1}1^p$ $|w|>p$, so must be "pump"-able
$|xy| \leq p$ so, $x=0^f y=0^g$, $f + g \leq p$ and $g>0$
Let's say $xy=0^p$ So $z=01^p$
When we pump $w$: $xy^2z$,
we get $0^p0^f0^g1^p \not\in A$
Let's try pumping down: $xy^2z$,
we get $xz \not\in A$

Number of 0s: $f+1$ Number of 1s: $p=f+g \geq f+1$
$f+1 \leq p$ number of 0s$<$number of 1 $xy^2z \not\in A$

Contradiction, pumped $w \not\in A$