CISC 4090
Theory of Computation

Context-Free Languages and Push Down Automata

Professor Daniel Leeds
dleeds@fordham.edu
JMH 332

Languages: Regular and Beyond

Regular:
- Captured by Regular Operations $(a \cup b) \cdot c^* \cdot (d \cup e)$
- Recognized by Finite State Machines

Context Free Grammars:
- Human language
- Parsing of computer language

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An example Context-Free Grammar

Grammar G1

$A \rightarrow 0A1$

$A \rightarrow B$

$B \rightarrow \#$

Example strings generated:
#, 0#1, 00#11, 000#111, …

$L(G1) = \{0^n\#1^n \mid n \geq 0\}$

Variables: A, B; Terminals: 0, 1, #

One start variable: A

Substitution rules/productions
- Variable -> Variables, Terminals

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Example English Grammar

Sentence -> NounPhrase VerbPhrase
NounPhrase -> Article NounSub
NounSub -> Noun | Adjective NounSub
VerbPhrase -> Verb | Verb NounPhrase
Noun -> Girl | Boy | Duck | Ball
Article -> The | A
Verb -> Throws | Sings

Example 1:
S -> NP VP
-> A NS V
-> A N V
-> The Boy Sings

Example 2:
S -> NP VP
-> A NS V
-> A N V
-> A Duck Throws

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Formal CFG Definition

A CFG is a 4-tuple (V, Σ, R, S)
- V is finite set of variables
- Σ finite set of terminals
- R finite set of rules
- $S \in V$ start variable

YetaAnother example

$G3 = (\{S\}, \{a, b\}, R, S)$
R: $S \to aSb \mid SS \mid \varepsilon$

Example strings generated:

L(G1) = { }

Another example

$G3 = (\{S\}, \{a, b\}, R, S)$
R: $S \to aSb \mid SS \mid \varepsilon$

Example rule expansion:

S -> aSb	S -> SS
aaSbb	aSb aSb
aaεbb	aεb aaSbb
aabb	aεb aaεbb
	abaabb

Example strings generated:
ε, ab, abab, aabb, aaabbbab, abababab, abaaabbb, …

L(G3) = {a's & b's; each a is followed by a matching b, every b matches exactly one corresponding preceding a}
(like parenthesis matching)

Another example

$G4 = (\{A, B, C\}, \{a, b, c\}, R, A)$
R: $A \to aA \mid BC \mid \varepsilon$
$B \to Bb \mid C$
$C \to c \mid \varepsilon$

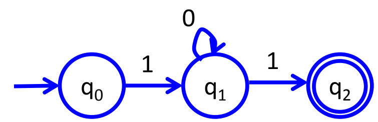
Example strings generated:

Designing CFGs

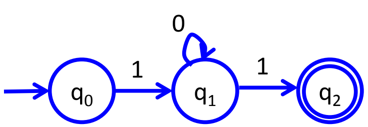
Creativity required

- If CFL is union of simpler CFL, design grammar for simpler ones (G1, G2, G3), then combine: S -> G1 | G2 | G3
- **If language is regular, can make CFG mimic DFA**

Example: express as CFG



Example: express as CFG



Q_0 -> $1Q_1$
Q_1 -> $0Q_1$ | $1Q_2$
Q_2 -> ε

Designing CFGs

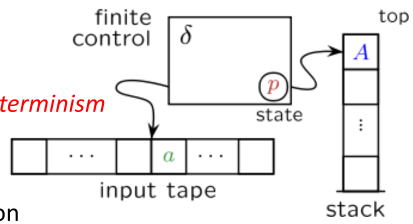
Creativity required

- **If language is regular, can make CFG mimic DFA**
  - Match each state with a single corresponding variable
    - $Q=\{q_0,\ldots,q_n\}$ $\quad V=\{R_0, \ldots, R_n\}$
  - Start state q_0 corresponds to state variable S -> R_0
  - Replace transition function with Production rule
    - $\delta(q_i, a) = q_j$ $\quad R_i \to aR_j$
  - Accept state q_k : transition to ε $\quad R_k \to \varepsilon$

Push down automata

FSA augmented with memory

Equivalent to CFG *if use non-determinism*



Finite control: transition function

Tape: holds input string

Stack: Can write to/read from stack

Input is Last In First Out ("LIFO")

PDA and Language 0^n1^n

Read symbol from input, push each 0 onto stack

As soon as see 1's, start popping 0 for each 1 seen

- If finish reading and stack empty, accept
- If stack is empty and 1's remain, reject
- If inputs finished but stack still has 0's, reject
- In 0 appears on input, reject

Definition of PDA

A PDA is a 6-tuple $(Q, \Sigma, \Gamma, \delta, q_0, F)$ where Q, Σ, Γ, and F are finite sets

- Q is sets of states
- Σ is the input alphabet
- Γ is the stack alphabet
- $\delta : Q \times \Sigma\varepsilon \times \Gamma\varepsilon \rightarrow P(Q \times \Gamma\varepsilon)$ is transition function
- $q_0 \in Q$ is start state
- $F \subseteq Q$ is set of accept states

PDA computation

M must start in q_0 with empty stack

M must move according to transition function

To accept string, M must be at accept state at end of input

Start stack with \$. If you see \$ at top of stack, it is empty

Understanding transition δ

$a, b \rightarrow c$ means:

- when you read a from tape and b is on top of stack
- replace b with c on top of stack

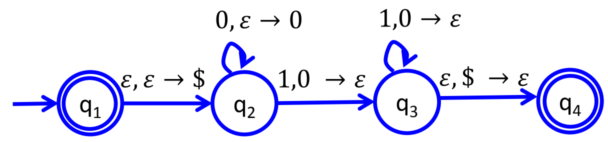
a, b, or c can be ε

- If a is ε then change stack without reading a symbol
- If b is ε then push new symbol c without popping b
- If c is ε then no new symbol pushed, only pop b

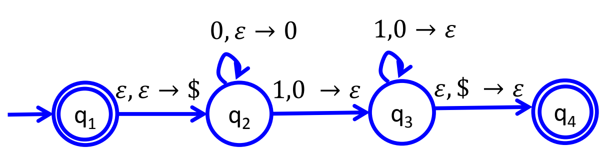
PDA to accept 0^n1^n

M1 is $(Q, \Sigma, \Gamma, \delta, q_0, F)$

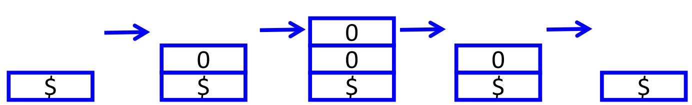
- $Q = \{q_1, q_2, q_3, q_4\}$ $\Sigma = \{0,1\}$
- $\Gamma = \{0, \$\}$ $F = \{q_1, q_4\}$



PDA to accept 0^n1^n



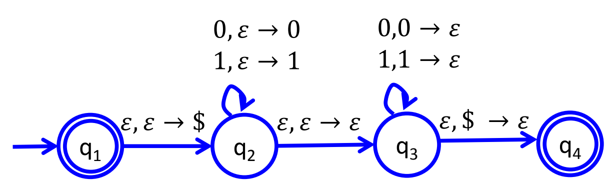
Input: 0011



PDA to accept $\{ww^R\}$

Power of non-determinism:

- At start, don't know where string w ends



PDA to accept $a^i b^j c^k$, i=j or j=k

Theorem: A language is context free if and only if some PDA recognizes it

Let's prove: If a language L is CFL, some PDA recognizes it

Idea: Show how CFG can define a PDA

- Stack has set of terminals/variables to compare with input
- Place proper terminal/variable pattern onto stack based on rules
- Non-determinism: Clone your machine, following different branches of rules

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CFG -> PDA

- If top of stack is variable, sub one right-hand rule for the variable
- If top of stack is terminal, keep going iff terminal matches input
- If top of stack is $, accept!

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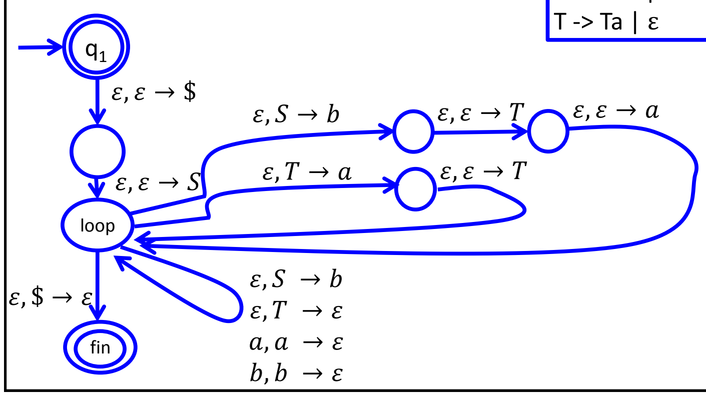
Example 2.25 in textbook

S -> aTb | b
T -> Ta | ε



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Chomsky Normal Form

CFG is in Chomsky normal form if every rule takes form:

$\mathrm{A} \rightarrow \mathrm{BC}$

$\mathrm{A} \rightarrow \mathrm{a}$

- B and C may not be the start variables
- The start variable may transition to ε

Any CFL can be generated by CFG in Chomsky Normal Form

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Converting to Chomsky Normal Form

- $S_0 \rightarrow S$ where S was original start variable
- Remove $A \rightarrow \varepsilon$
- Shortcut all unit rules
  Given $A \rightarrow B$ and $B \rightarrow u$, add $A \rightarrow u$
- Replace variable-terminal rules with variable-variable rules
  Given $A \rightarrow Bc$, add $U_C \rightarrow c$ and change A to $A \rightarrow BU_C$
- Replace rules $A \rightarrow u_1u_2u_3 \dots u_k$ with:
  $A \rightarrow u_1A_1,\ A_1 \rightarrow u_2A_2,\ A_2 \rightarrow u_3A_3,\ \dots,\ A_{k-2} \rightarrow u_{k-1}u_k$

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Conversion practice

Non-normal form:

$S \rightarrow aSa|bX$

$X \rightarrow Ycc|\varepsilon$

$Y \rightarrow d|c$

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Ambiguity – examples

A grammar may generate a string in multiple ways

Math example:

$\mathrm{Expr} \rightarrow \mathrm{Expr} + \mathrm{Expr} \mid \mathrm{Expr} \times \mathrm{Expr} \mid \mathrm{Expr} \mid \mathrm{a}$

English example:

the girl touches the boy with the flower

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Ambiguity – definitions

A grammar generates a string ambiguously if there are two or more different parse trees

Definitions:

- Leftmost derivation: at each step the leftmost remaining variable is replaced
- *w* is derived **ambiguously** in CFG G if there exist more than one leftmost derivations

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Conversion practice

Non-normal form:

$$S \rightarrow aa|bXc$$
$$X \rightarrow Xc|Y$$
$$Y \rightarrow Ycc|a$$

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Regular languages vs. CFLs

- CFGs define CFLs
- PDAs recognize CFLs and Regular languages
- FSAs recognize Regular languages, but **not** CFLs
- CFLs and Regular languages not equivalent

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Non Context Free Languages

Languages recognized by PDAs

- $L=\{ww^R\}$
- $L=\{a^nb^n \mid n\geq 0\}$

Languages **not** recognized by PDAs

- $L=\{ww\}$
- $L=\{a^nb^nc^n \mid n\geq 0\}$

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Proving non context free – NEW pumping lemma!

Every string in CFL A with length greater than or equal to the pumping length *p* can be "pumped"

Every string $w \in A$ ($|w| \geq p$) can be written as uvxyz where

1. For each $i \geq 0$, $uv^i xy^i z \in A$
2. $|vy| > 0$
3. $|vxy| \leq p$


Regular language PUMPING: Proof idea

If $|s| < p$, trivially true

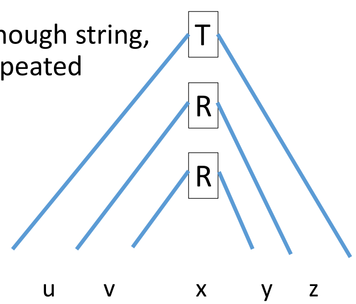
If $|s| \geq p$, consider the states the FSA goes through

- Since there are only p states, $|s|>p$, one state must be repeated
- **Pigeonhole principle:** There must be a cycle

CFL pumping: Proof idea

Pigeonhole idea: Given a long enough string, some variable will need to be repeated

Example Grammar: S -> uRz
R -> x | vRy



Prove F={ww | $w=(0 \cup 1)^*$} not CFL

Try a sample string $s=\{0^p10^p1\}$ $|s|>p$

- Can we define uvxyz=s so $uv^ixy^iz \in F$?
- Yes: $u=0^{p-1}$, $v=0$, $x=1$, $y=0$, $z=0^{p-1}1$

Try another sample string $s=\{0^p1^p0^p1^p\}$

- Can we define uvxyz=s so $uv^ixy^iz \in F$?
- No:
  - If vxy is in first w, pumping will make increase 1's and/or 0's in first w but not in second
  - If vxy straddles the middle, vxy will either increase 1's for first w and 0's for second w, or will break the 0^n1^n pattern

Prove $B=\{a^n b^n c^n \mid n \geq 0\}$ not CFL

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