# CISC 4090
# Theory of Computation

Complexity

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## Computability

Are we guaranteed to get an answer?

## Complexity

How long do we have to wait for an answer? (Ch7)

How much resources do we need to find the answer? (Ch8)

## Time complexity example: Deciding $\{0^k1^k | k \geq 0\}$

How much time does a TM take to decide $A=\{0^k1^k | k \geq 0\}$?

Computation steps:

- Scan left to right and confirm no 0's after 1's
- Repeat if both 0s and 1 left on tape
  - Scan across tape removing a single 0 and a single 1
- If neither 0's or 1's remain on tape, accept; otherwise, reject

## Characterizing run-time

If TM M halts on all inputs, there exists f: N->N
f(n) = max # steps on any input of length n

- "M runs in time f(n)"
- "M is an f(n) time Turing machine"

## Asymptotic analysis – “Big O” and “Small O”

Assess runtime as input grows large

- Only consider highest-order term
- Ignore constant co-efficients

Example: $f(n) = 5n^4+3n^2+10n+5$

$f(n)=O(???)$

## Big-O Defined

$f(n) = O(g(n))$ if positive integers *c* and $n_0$ exist such that for every $n \geq n_0$

- $f(n) \leq c\, g(n)$
- $g(n)$ is “asymptotic upper bound” for $f(n)$

Big-O does not require the upper bound to be “tight”

## Beyond polynomial

Exponential bounds, like $O(2^n)$, much bigger

Logarithmic bounds, like $O(\log n)$, much smaller

$O(\log_2 n) = O(\log_{10} n) = O(\ln n)$ – only constant difference

## Small-O defined

$f(n) = o(g(n))$ if for any real number $c>0$, $n_0$ exists such that for every $n \geq n_0$

- $f(n) < c\, g(n)$

In other words:

$f(n) = o(g(n))$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$

## Time complexity example: Deciding $\{0^k1^k | k \geq 0\}$

How much time does a TM take to decide $A=\{0^k1^k | k \geq 0\}$?

Computation steps:

- Scan left to right and confirm no 0's after 1's
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## Time complexity example

Computation steps:

- Scan left to right and confirm no 0's after 1's **O(n)**
- Repeat if both 0s and 1 left on tape **$O\left(\frac{n}{2}\right)$**
  - Scan across tape removing a single 0 and a single 1 **O(n)**
- If neither 0's or 1's remain on tape, accept; otherwise, reject **O(n)**

Total complexity: $O(n)+O(n/2)O(n)+O(n)+O(n) = 3\,O(n) + O(n)O(n)$

$= 3\,O(n) + O(n^2) = \mathbf{O(n^2)}$

## Time complexity class

Let $t: N \to R^+$ be a function

TIME(t(n)) is collection of all languages decidable by an O(t(n)) time Turing machine

TIME(t(n)) is a time complexity class

## Tighter bound for $\{0^k1^k\}$ on TM

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Scan across tape, reject if 0 found right of 1

Repeat as long as some 0s and 1s remain

- Scan across tape and confirm total number of 0s and 1s is even
- Scan again, crossing off every other 0 starting with the first 0 and every other 1, starting with first 1

If no 0s and no 1s left, accept; else reject

## Computing the tighter bound

Each step cuts input size in half.
Number of times to cut n in half: $\log_2 n$

Scan across tape, reject if 0 found right of 1 O(n)

Repeat as long as some 0s and 1s remain O($\log_2$ n)

- Scan across tape and confirm total number of 0s and 1s is even O(n)
- Scan again, crossing off every other 0 starting with the first 0 and every other 1, starting with first 1 O(n)

If no 0s and no 1s left, accept; else reject O(n)

O(n) + O($\log_2$n) (O(n)+O(n)) + O(n) = O(n) + O($\log_2$n) O(n)
= **O(n log n)**

## Complexity on 2-tape Turing machine

Scan across tape and reject if 0 right of 1

Scan across 0s on tape 1, writing each onto tape 2

Scan across the 1s on tape 1, removing a 0 from tape 2 for each 1

If run out of 0s while still reading 1s, reject

If 0s left after 1s finished, reject

If no 0s left, accept

## Complexity on 2-tape Turing machine **computed**

Scan across tape and reject if 0 right of 1 O(n)

Scan across 0s on tape 1, writing each onto tape 2 O(n)

Scan across the 1s on tape 1, removing a 0 from tape 2 for each 1 O(n)

If run out of 0s while still reading 1s, reject O(n)

If 0s left after 1s finished, reject

If no 0s left, accept

4 O(n) = **O(n)**

## Relationship between Single and Multi-tape TM

Theorem: Let t(n) be function, where $t(n) \geq n$. Then every t(n) time multitape TM has equivalent O($t^2$(n)) time single-tape TM

Convert any multi-tape TM to single-tape TM

Active part of tape t(n); make t(n) traversals through tape: $t^2(n)$

## Relationship between Single and Multi-tape TM

Theorem: Let t(n) be function, where $t(n) \geq n$. Then every t(n) time multitape TM has equivalent $O(t^2(n))$ time single-tape TM

Convert any multi-tape TM M to single-tape TM S

Across full runtime, M makes visits at most t(n) tape locations

In S simulation, include t(n) locations from each of the *k* tapes

For each S step, need to read/write each tape – k t(n) steps to traverse all tapes in one direction

S takes t(n) computation loops, one for each step of M

$O(t(n))\ O(t(n)) = \mathbf{O(t^2(n))}$

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## Relationship between DTM and NDTM

Let N be NDTM that is a decider. Run time of N is max number of steps that N uses on any branch of its computation on input of length n

Does not correspond to real-world computer

- Except maybe a quantum computer!

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## NDTM -> DTM

If NDTM N decides language A in t(n) steps, can construct DTM D to decide A in $O(b^{t(n)})$ steps, where b is the maximum number of possible branches for a state-input pair.

D must simulate each branch of N, using breadth first search

At each step of computation, N chooses one branch down tree of possible branches

If b possible branches taken at each step, and there are t(n) steps, there are a total of $b^{t(n)}$ possible terminal points – exponential!!!

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## Polynomial time

Difference between polynomial times considered small compared to exponential time

Big picture:

- Exponential: brute force trying every solution to see what fits
- Polynomial: more efficient computation

"Reasonable" computational models are polynomial-time equivalent

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## Class P

P is the class of languages decidable in polynomial time on a deterministic single-tape Turing Machine

$$\mathrm{P} = \cup_k \ \mathrm{TIME}(\mathrm{n}^{\mathrm{k}})$$

P is class of problems realistically solvable on a computer*

## Problems in P

To show an algorithm belongs to P, we need to:
- Provide polynomial upper bound on number of stages
- Examine each stage to ensure it can run in polynomial time

Polynomial composed with polynomial -> polynomial

Address input encoding size:
- Graphs as list of nodes and edges, or adjacency matrix
- Binary encoding of integers

## Example: Path problem

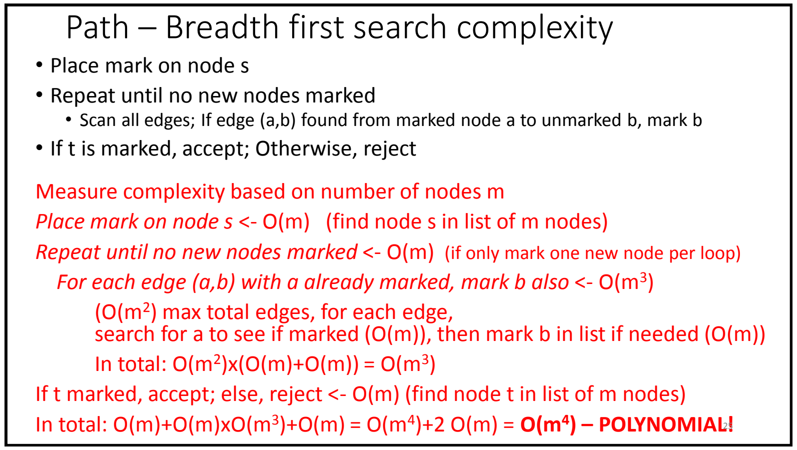
Is there a path from s to t in graph G?
- PATH={<G,s,t> | G is directed graph with directed path from s to t}

Brute force
- If G has m nodes, path cannot be more than m
- Upper bound on possible paths $m^m$
- Try each “path” one by one for legality and for linking s-to-t

## Example: Path problem

Is there a path from s to t in graph G?
- PATH={<G,s,t> | G is directed graph with directed path from s to t}

Breadth first search
- Place mark on node s
- Repeat until no new nodes marked
  - Scan all edges; If edge (a,b) found from marked node a to unmarked b, mark b
- If t is marked, accept; Otherwise, reject



## Path – Breadth first search complexity

- Place mark on node s
- Repeat until no new nodes marked
  - Scan all edges; If edge (a,b) found from marked node a to unmarked b, mark b
- If t is marked, accept; Otherwise, reject

Measure complexity based on number of nodes m

*Place mark on node s* <- O(m) (find node s in list of m nodes)

*Repeat until no new nodes marked* <- O(m) (if only mark one new node per loop)

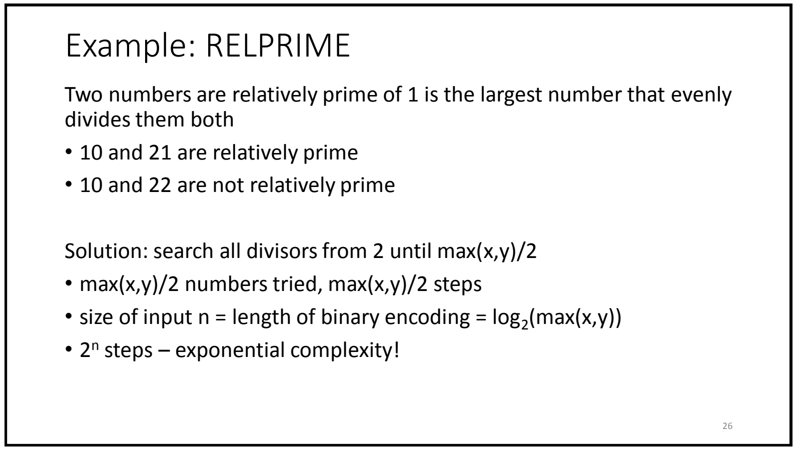
*For each edge (a,b) with a already marked, mark b also* <- O($m^3$)

($O(m^2)$) max total edges, for each edge,
search for a to see if marked (O(m)), then mark b in list if needed (O(m))

In total: $O(m^2)$x(O(m)+O(m)) = $O(m^3)$

If t marked, accept; else, reject <- O(m) (find node t in list of m nodes)

In total: O(m)+O(m)x$O(m^3)$+O(m) = $O(m^4)$+2 O(m) = **$O(m^4)$ – POLYNOMIAL!**



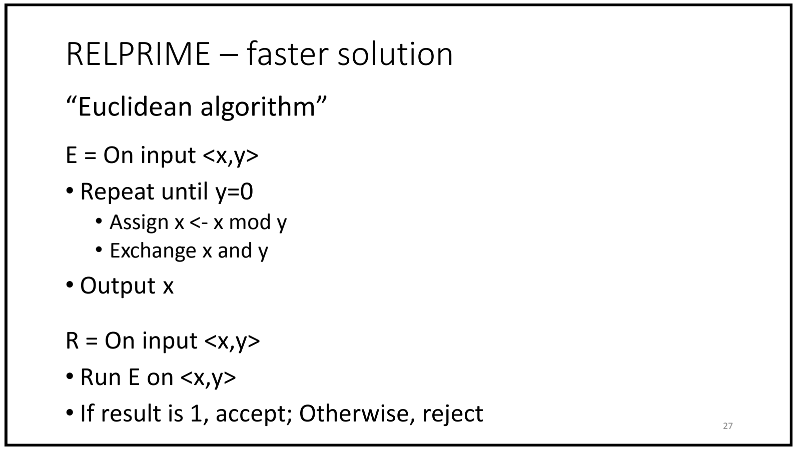
## Example: RELPRIME

Two numbers are relatively prime of 1 is the largest number that evenly divides them both

- 10 and 21 are relatively prime
- 10 and 22 are not relatively prime

Solution: search all divisors from 2 until max(x,y)/2

- max(x,y)/2 numbers tried, max(x,y)/2 steps
- size of input n = length of binary encoding = $\log_2$(max(x,y))
- $2^n$ steps – exponential complexity!



## RELPRIME – faster solution

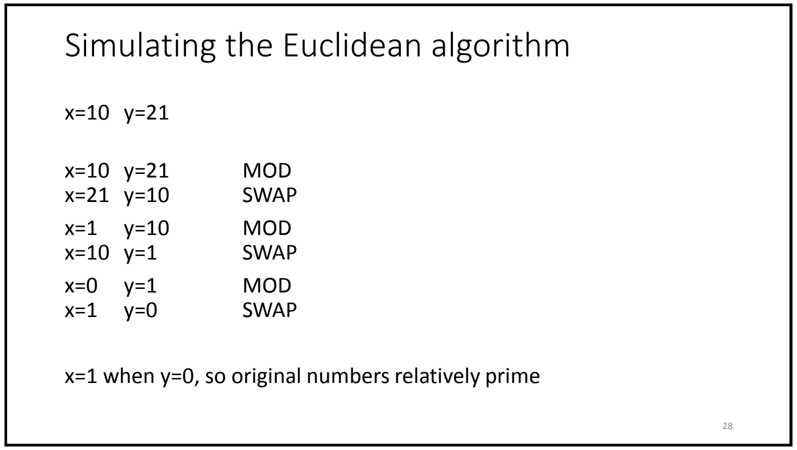
"Euclidean algorithm"

E = On input <x,y>

- Repeat until y=0
  - Assign x <- x mod y
  - Exchange x and y
- Output x

R = On input <x,y>

- Run E on <x,y>
- If result is 1, accept; Otherwise, reject



## Simulating the Euclidean algorithm

x=10 y=21

| | | |
|---|---|---|
| x=10 | y=21 | MOD |
| x=21 | y=10 | SWAP |
| x=1 | y=10 | MOD |
| x=10 | y=1 | SWAP |
| x=0 | y=1 | MOD |
| x=1 | y=0 | SWAP |

x=1 when y=0, so original numbers relatively prime

## Euclidean Algorithm – complexity

x = x mod y <- new x always less than y

- If old x is twice y or more, new x will be cut at least in half
- If old x between y and 2y, new x will be cut at least in half
  - new x = x-y

Number of loops: $2\log_2(\max(x,y))$
Length of input (in binary): $\log_2(x)+\log_2(y)=O(\log_2(\max(x,y)))$
Number of loops: O(n)

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## Hamiltonian Path Example

Hamiltonian path in directed graph G: directed path that goes through each node exactly once

HAMPATH = { <G,s,t> | G is directed graph with Hamiltonian path from s to t}

Brute force
- List all possible paths and confirm if it's a valid path
- If m nodes, m! paths – exponential

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## Polynomial Verifiability – e.g., HAMPATH

- If given an answer, can determine if it is correct in polytime
- Path at most m long, graph has at most $m^2$ edges

For each proposed edge in path O(m)
- Search list of edges in G to find/not-find path edge $O(m^2)$

Total complexity: $O(m^3)$

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## Verifier definition

Verifier for language A is algorithm V where:
- A={w | V accepts <w,c> for some string c}
- Verifiers uses extra information in the "certificate" c to verify string w is in A

A is polytime verifiable if it has a polytime verifier
- Complexity measured in terms of w

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## NP definition

NP is class of languages that have polytime verifiers

NP <- Nondeterministic Polynomial time

HAMPATH c: proposed m-1 step path along nodes in graph G.

For HAMPATH

- Write list of m nodes, nondeterministically selected
- Check for repeats in list; if repeats, reject
- Check if s=$node_1$ and t=$node_m$; if either fails, reject
- For each i between 1 and m-1, check ($node_i$,$node_{i+1}$) is in G. If not, reject; else except.

## NTIME and Class NP

NTIME(t(n)) = {L | L is language decided by O(t(n)) time nondeterministic Turing machine}

$$\mathrm{NP} = \cup_k \ \mathrm{NTIME}(\mathrm{n}^{\mathrm{k}})$$

## NP Example: CLIQUE

A clique in an undirected graph is a subgraph where every two nodes are connected by an edge

- A k-clique is a clique containing k nodes

CLIQUE: {<G,k> | G is an undirected graph with a k-clique}

- k is a parameter. Determining clique for certain fixed k is in P

## Prove CLIQUE∈NP

Just prove clique is certificate

Here is a verifier for V for CLIQUE

- V = On input <<G,k>, c> (m nodes)
  - Test if c contains k nodes, all in G O(m)
  - Test whether G contains all edges connecting all nodes in c
    *For each node pair in c, check if matching pair in G*
    O($m^2$)xO($m^2$) -> O($m^4$)
  - If both pass, accept; else, reject

This is a polytime algorithm! O($m^4$)

NTM method:

- Nondeterministically generate c
- Test c with V

## Another NP Example: Subset-Sum

- SUBSET-SUM problem: given a list of numbers S={$x_1$,$x_2$,…,$x_n$}, determine if a subset of numbers add to the target t
- SUBSET-SUM={<S,t>|S={$x_1$,…,$x_n$} and some {$y_1$,…,$y_k$}⊆{$x_1$,…,$x_n$} and $\sum y_i = t$}

SUBSET-SUM ∈ NP

Polytime verifiable!

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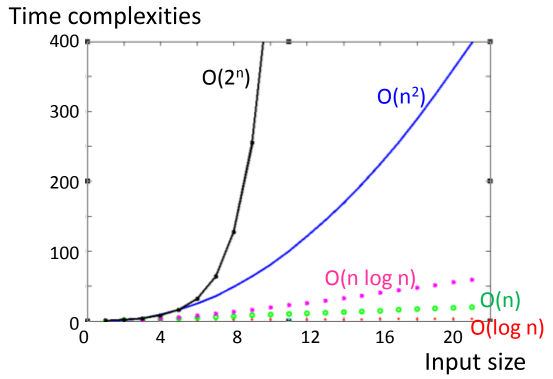

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## Does P = NP?

*In other words: Are there polynomial time solutions to all algorithms that are polytime verifiable?*

Probably not, but it’s an open question!

If P=NP, lots of “hard” problems become doable – including cracking encrypted networks!

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## Class coNP

- A language is coNP if its complement is NP
- Unknown if all coNP languages are also NP

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## NP-Complete problems

The <u>hardest</u> problems in NP are **NP complete**

If polynomial time algorithm for these problems, P=NP

- If any NP problem requires more than polytime, then NP-complete problems also require more than polytime

Since no polytime solution has been found for an NP-complete problem, if we determine new problem is NP-complete, reasonable to give up search for general polytime solution to this problem

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## Satisfiability: An NP-complete problem

Consider Boolean operator AND, OR, NOT

Consider set of Boolean variables

Boolean formula is satisfiable if some assignment of T's and F's makes the total formula True (T)

- Ex: $(x' \vee y) \wedge (x \vee z')$
- Ex: $(x' \vee y) \wedge (x \vee y') \wedge (x \vee y) \wedge (x' \vee y')$

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## Cook-Levin Theorem

SAT can be used to solve all problems in NP

I.e., SAT∈P iff P=NP

I.e, every problem in NP **efficiently reduces** to SAT

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## Polynomial time reducibility

When problem A is efficiently reducible to problem B, an efficient solution to B will efficiently solve A

- **"Efficiently reducible"** means in polynomial time
- If language A is polytime reducible to language B, and B has polytime solution, then A has polytime solution!
- A efficiently reduces to B, B efficiently reduces to C

  -> A efficiently reduces to C

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## 3SAT

Special case of satistfiability

Terms:

- Boolean or negated Boolean is "literal"
- Clause is several literals joined by V
- Boolean formula is in "conjunctive normal form" if it has only clauses connected by Λs

"3cnf" – each clause has 3 literals

- E.g., (x1Vx2Vx3) Λ (x1Vx2'Vx3) Λ (x1'Vx2Vx3)

- 3SAT – language of 3cnf formulas that are satisfiable

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## Polytime reduction from 3SAT to CLIQUE

Given 3SAT formula create a graph G

Nodes in G are organized into k groups of 3 nodes called $t_1,\ldots,t_k$

Each triple corresponds to one of the clauses in the formula

Each node in triple is labeled with literal in the clause

Edges in G connect all but two types of pairs of nodes in G

- No edge is between nodes in same triple
- No edge is present between nodes with contradictory labels

Formula satisfiable iff G has k-clique

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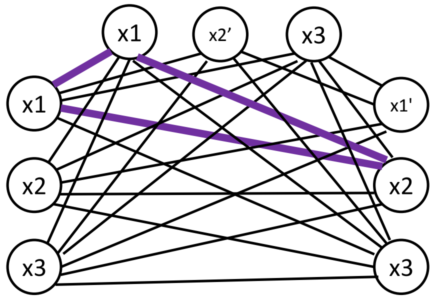
## (x1Vx2Vx3) Λ (x1Vx2'Vx3) Λ (x1'Vx2Vx3)



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## Why does this reduction work?

At least one value must be true per clause

Select one node corresponding to true literal in satisfying assignment

Selected nodes must form k-clique

- There will be k nodes
- Every pair of nodes will be connected by an edge because they don't violate one of the 2 no-edge conditions

Alternatively

- If k-clique, all literals are in different clauses and hence will be satisfiable, since any logically inconsistent assignments are not represented in G

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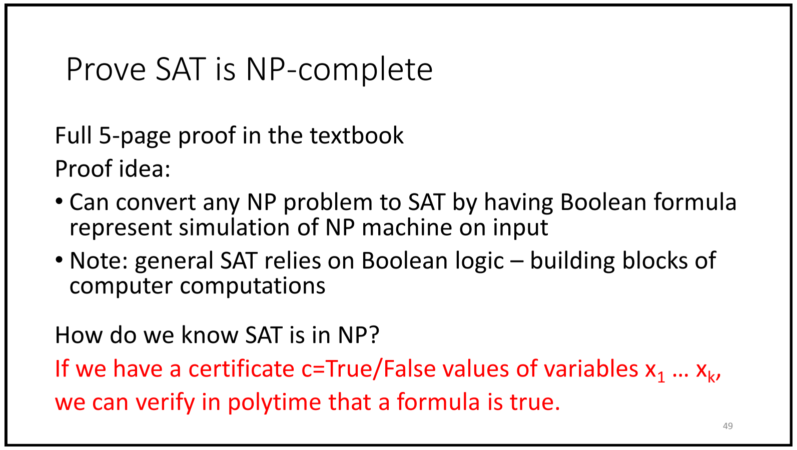

## Prove SAT is NP-complete

Full 5-page proof in the textbook

Proof idea:

- Can convert any NP problem to SAT by having Boolean formula represent simulation of NP machine on input
- Note: general SAT relies on Boolean logic – building blocks of computer computations

How do we know SAT is in NP?

If we have a certificate c=True/False values of variables $x_1$ … $x_k$, we can verify in polytime that a formula is true.

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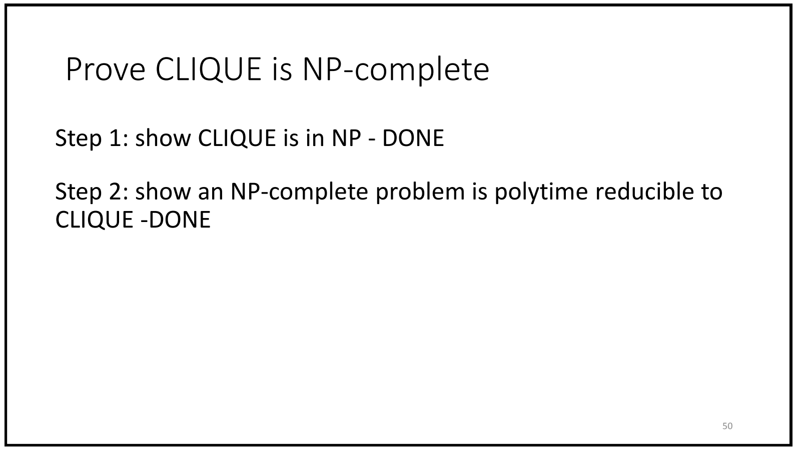

## Prove CLIQUE is NP-complete

Step 1: show CLIQUE is in NP - DONE

Step 2: show an NP-complete problem is polytime reducible to CLIQUE -DONE

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