# Dimensionality reduction

CISC 5800

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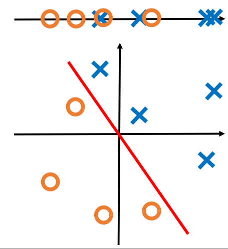
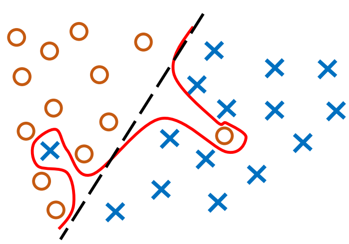
## The benefits of extra dimensions



- **Finds existing complex separations between classes**



## The risks of too-many dimensions



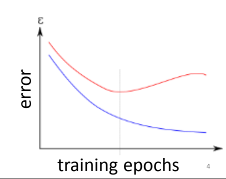
- **High dimensions with kernels over-fit the outlier data**
- Two dimensions ignore the outlier data



## Training vs. testing

- **Training**: learn parameters from set of data in each class
- **Testing**: measure how often classifier correctly identifies new data

- More training reduces classifier error $\varepsilon$
  - More gradient ascent steps
  - More learned feature
- Too much training causes worse testing error – overfitting






## Goal: High Performance, Few Parameters

- "Information criterion": performance/parameter trade-off
- Variables to consider:
  - **L** likelihood of train data after learning
  - **k** number of parameters (e.g., number of features)
  - **m** number of points of training data
- Popular information criteria:
  - Akaike information criterion **AIC**: log(L) - k
  - Bayesian information criterion **BIC**: log(L) - 0.5 k log(m)



## Decreasing parameters

- Force parameter values to 0
  - L1 regularization
  - Support Vector selection
  - Feature selection/removal
- Consolidate feature space
  - Component analysis

## Feature removal

- Start with feature set: $F=\{x_1, \ldots, x_k\}$
- Find classifier performance with set F: perform(F)
- Loop
  - Find classifier performance for removing feature $x_1, x_2, \ldots, x_k$: $\text{argmax}_i$ perform($F-x_1$)
  - Remove feature that causes least decrease in performance: $F=F-x_i$

Repeat, using AIC or BIC as termination criterion

**AIC**: log(L) - k

**BIC**: log(L) - 0.5 k log(m)

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## AIC testing: log(L)-k

| Features | k (num features) | L (likelihood) | AIC |
|---|---|---|---|
| F | 40 | 0.1 | -42.3 |
| F-$\{x_3\}$ | 39 | 0.03 | -41.5 |
| F-$\{x_3,x_{24}\}$ | 38 | 0.005 | -41.3 |
| F-$\{x_3,x_{24},x_{32}\}$ | 37 | 0.001 | -40.9 |
| F-$\{x_3,x_{24},x_{32},x_{15}\}$ | 36 | 0.0001 | **-41.2** |

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## Feature selection

- Find classifier performance for just set of 1 feature: $\text{argmax}_i$ perform($\{x_i\}$)
- Add feature with highest performance: $F=\{x_i\}$
- Loop
  - Find classifier performance for adding one new feature: $\text{argmax}_i$ perform($F+\{x_i\}$)
  - Add to F feature with highest performance increase: $F=F+\{x_i\}$

Repeat, using AIC or BIC as termination criterion

**AIC**: log(L) - k

**BIC**: log(L) - 0.5 k log(m)

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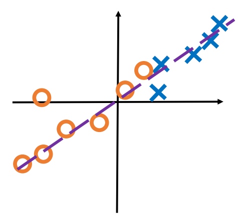
## Defining new feature axes



- Identify a common trend

$$\boldsymbol{u_1} = \begin{bmatrix} 0.91 \\ 0.45 \end{bmatrix}$$

- Map data onto new dimension $\boldsymbol{u_1}$

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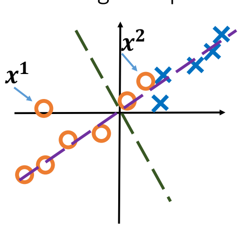
## Defining data points with new axes




$$\boldsymbol{x^1} = -1 \times \boldsymbol{u_1} + (-0.5) \times \boldsymbol{u_2}$$

$$\boldsymbol{x^2} = \boldsymbol{0.5} \times \boldsymbol{u_1} + (-0.1) \times \boldsymbol{u_2}$$

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## Component analysis

Each data point $\boldsymbol{x^i}$ in D can be reconstructed as sum of components $\boldsymbol{u}$:

- $\boldsymbol{x^i} = \sum_{q=1}^{T} z_q^i \boldsymbol{u_q}$
- $z_q^i$ is weight on q[th] component to reconstruct data point $\mathbf{x^i}$

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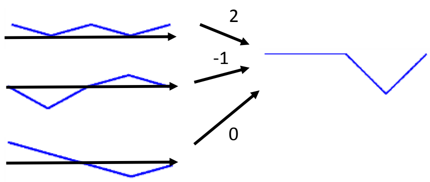
## Component analysis: examples

Components | Data






## Component analysis: examples

"Eigenfaces" – learned from set of face images

**u**: nine components

**x**$^i$: data reconstructed



## Types of component analysis

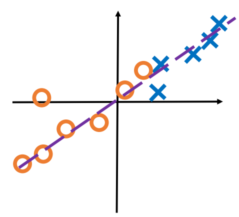
$$x^i = \sum_{q=1}^{T} z_q^i \boldsymbol{u_q}$$

Learn new axes from data sets: common "components"

- Principal component analysis (PCA):
  - Best reconstruction of each data point $\boldsymbol{x}^i$ with first $t$ components
  - Each component perpendicular to all others: $(\boldsymbol{u_i})^T \boldsymbol{u_j} = 0 \quad \forall i \neq j$
- Independent component analysis (ICA):
  - Minimize number of components to describe each $x^i$
  - Can focus on different components for different $x^i$
- Non-negative matrix factorization (NMF):
  - All data $\boldsymbol{x}^i$ non-negative
  - All components and weights non-negative $\boldsymbol{u}_j \geq 0,\ z_q^i \geq 0 \quad \forall i, q$



## Principle component analysis (PCA)



Start with

- D = {$\boldsymbol{x^1}$,…,$\boldsymbol{x^n}$} , data 0-center
- Component index: q=1

Loop

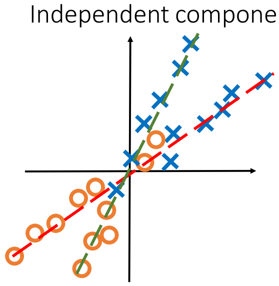
- Find direction of highest variance: $\boldsymbol{u_q}$
  - Ensure $|\boldsymbol{u_q}| = 1$
- Remove $\mathbf{u_q}$ from data: $D = \{\boldsymbol{x^1} - z_q^1\, \boldsymbol{u_q}, \cdots, \boldsymbol{x^n} - z_q^n \boldsymbol{u_q}\}$

We **require** $(\boldsymbol{u_i})^T \boldsymbol{u_j} = 0 \quad \forall i \neq j$

Thus, we guarantee $z_j^i = \boldsymbol{u}_j^T \boldsymbol{x}^i$



## Independent component analysis (ICA)



Start with

- D = {$\boldsymbol{x^1}$,…,$\boldsymbol{x^n}$} , data 0-center

Find group(s) for each data point

Find direction for each group $\boldsymbol{u_q}$

- Ensure $|\boldsymbol{u_q}| = 1$

We do **not** require $(\boldsymbol{u_i})^T \boldsymbol{u_j} = 0 \quad \forall i \neq j$

Thus, we cannot guarantee $z_j^i = \boldsymbol{u}_j^T \boldsymbol{x}^i$



## Evaluating components

- Components learned in order of descriptive power
- Compute reconstruction error for all data by using first v components:

$$error = \sum_i \left( \sum_j \left( x_j^i - \sum_{q=1}^{v} a_q^i \boldsymbol{u}_{q,j} \right)^2 \right)$$