Discrete Structure II: Introduction

Outline

- Overview of the course
- Getting started, “speaking mathematically”
  - variables
  - universal, existential and conditional statements
  - set
  - relations and functions

Course Organization

- What is the purpose of lab sections?
  - really problem solving sections, or recitations
  - working on problems (paper and pencils) under the guidance of the instructor, sometimes in groups
  - VERY important for your successful mastery of concepts/methods/skills taught in lectures.
- Syllabus
  - Web site: http://storm.cis.fordham.edu/zhang/cs2100
  - Email is the best way to reach me
  - Office hours are open (just stop by)
  - Assessment: academic integrity

Course Overview

- a more in-depth, rigorous study of following mathematical subjects that are important to computer science
  - Logic:
    - arguments, digital logic circuits,
    - quantified statements
  - Proof: how to construct a carefully reasoned argument to convince someone that a given statement is true
    - various methods: direct proof, proof by contradiction…
  - Mathematical induction: a powerful proof technique
    - Example: can we replace pennies with 3 cent coins?
    - Correctness of algorithms: reasoning about loops

Course Overview (2)

- Recursion: recursively defined sequence and sets, recursive function
  - ex: recurrence relation, e.g., $P_n=P_{n-1}+P_{n-2}$
  - Recursive algorithms, e.g., Tower of Hanoi, merge sort, …
- Set theory: go beyond basics,
  - Halting problem (Alan Turing): something that computers cannot do… (to be revisited in Theory of Computation)
- Functions:
  - cardinality and computability
  - e.g., Are there more rationals than integers?

Course Overview (3)

- Relations:
  - modular arithmetic and cryptography
  - PERT and CPM
- Counting and Probability
  - Monty Hall Problem
  - Bayes' Theorem, …
- Analysis of algorithm efficiency
  - e.g., running time of binary search algorithm, merge sort algorithm? …

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Variables

There are two uses of a variable. To illustrate the first use, consider asking

Is there a number with the following property: doubling it and adding 3 gives the same result as squaring it?

In this sentence you can introduce a variable to replace the potentially ambiguous word "it":

Is there a number x with the property that $2x + 3 = x^2$?

Variables

The advantage of using a variable is that it allows you to give a temporary name to what you are seeking so that you can perform concrete computations with it to help discover its possible values.

To illustrate the second use of variables, consider the statement:

No matter what number might be chosen, if it is greater than 2, then its square is greater than 4.

Variables

In this case introducing a variable to give a temporary name to the (arbitrary) number you might choose enables you to maintain the generality of the statement, and replacing all instances of the word “it” by the name of the variable ensures that possible ambiguity is avoided:

No matter what number n might be chosen, if n is greater than 2, then n^2 is greater than 4.

Example 1 – *Writing Sentences Using Variables*

Use variables to rewrite the following sentences more formally.

a. Are there numbers with the property that the sum of their squares equals the square of their sum?

b. Given any real number, its square is nonnegative.

Solution:

a. Are there numbers a and b with the property that $a^2 + b^2 = (a + b)^2$?

Or: Are there numbers a and b such that $a^2 + b^2 = (a + b)^2$?

Example 1 – *Solution*

cont'd

Or: Do there exist any numbers a and b such that $a^2 + b^2 = (a + b)^2$?

b. Given any real number r, r^2 is nonnegative.

Or: For any real number r, $r^2 \geq 0$.
Or: For all real numbers r, $r^2 \geq 0$.

Some Important Kinds of Mathematical Statements

Some Important Kinds of Mathematical Statements

Three of the most important kinds of sentences in mathematics are universal statements, conditional statements, and existential statements:

A **universal statement** says that a certain property is true for all elements in a set. (For example: *All positive numbers are greater than zero.*)

A **conditional statement** says that if one thing is true then some other thing also has to be true. (For example: *If 378 is divisible by 18, then 378 is divisible by 6.*)

Given a property that may or may not be true, an **existential statement** says that there is at least one thing for which the property is true. (For example: *There is a prime number that is even.*)

Some Important Kinds of Mathematical Statements

Universal Condition Statements

Universal statements contain some variation of the words "for all" and conditional statements contain versions of the words "if-then."

Some Important Kinds of Mathematical Statements

A ***universal conditional statement*** is a statement that is both universal and conditional. Here is an example:

For all animals *a*, if *a* is a dog, then *a* is a mammal.

One of the most important facts about universal conditional statements is that they can be rewritten in ways that make them appear to be purely universal or purely conditional.

Example 2 – *Rewriting an Universal Conditional Statement*

Fill in the blanks to rewrite the following statement:
For all real numbers x, if x is nonzero then x^2 is positive.

a. If a real number is nonzero, then its square _____.

b. For all nonzero real numbers x, ____.

c. If x ____, then ____.

d. The square of any nonzero real number is ____.

e. All nonzero real numbers have ____.

Example 2 – *Solution*

a. is positive

b. x^2 is positive

c. is a nonzero real number; x^2 is positive

d. Positive

e. positive squares (*or*: squares that are positive)

Some Important Kinds of Mathematical Statements

Universal Existential Statements

A ***universal existential statement*** is a statement that is universal because its first part says that a certain property is true for all objects of a given type, and it is existential because its second part asserts the existence of something. For example:

Every real number has an additive inverse.

In this statement the property "has an additive inverse" applies universally to all real numbers.

Some Important Kinds of Mathematical Statements

"Has an additive inverse" asserts the existence of something—an additive inverse—for each real number.

However, the nature of the additive inverse depends on the real number; different real numbers have different additive inverses.

Example 3 – *Rewriting an Universal Existential Statement*

Fill in the blanks to rewrite the following statement:
Every pot has a lid.

a. All pots ______.

b. For all pots *P*, there is _____.

c. For all pots *P*, there is a lid *L* such that ______.

Solution:

a. have lids

b. a lid for *P*

c. *L* is a lid for *P*

Some Important Kinds of Mathematical Statements

Existential Universal Statements

An ***existential universal statement*** is a statement that is existential because its first part asserts that a certain object exists and is universal because its second part says that the object satisfies a certain property for all things of a certain kind.

Some Important Kinds of Mathematical Statements

For example:

There is a positive integer that is less than or equal to every positive integer:

This statement is true because the number one is a positive integer, and it satisfies the property of being less than or equal to every positive integer.

Example 4 – *Rewriting an Existential Universal Statement*

Fill in the blanks to rewrite the following statement in three different ways:

There is a person in my class who is at least as old as every person in my class.

a. Some _____ is at least as old as _____.

b. There is a person p in my class such that p is _____.

c. There is a person p in my class with the property that for every person q in my class, p is _____.

Example 4 – *Solution*

a. person in my class; every person in my class

b. at least as old as every person in my class

c. at least as old as q

Some Important Kinds of Mathematical Statements

Some of the most important mathematical concepts, such as the definition of limit of a sequence, can only be defined using phrases that are universal, existential, and conditional, and they require the use of all three phrases “for all,” “there is,” and “if-then.”

Some Important Kinds of Mathematical Statements

For example, if $a_1, a_2, a_3, \ldots$ is a sequence of real numbers, saying that

the limit of a_n as n approaches infinity is L

means that

for all positive real numbers ε, **there is** an integer N such that **for all** integers n, **if** $n > N$ **then** $-\varepsilon < a_n - L < \varepsilon$.

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The Language of Sets

Use of the word *set* as a formal mathematical term was introduced in 1879 by Georg Cantor (1845–1918). For most mathematical purposes we can think of a set intuitively, as Cantor did, simply as a collection of elements.

For instance, if C is the set of all countries that are currently in the United Nations, then the United States is an element of C, and if I is the set of all integers from 1 to 100, then the number 57 is an element of I.

The Language of Sets

- **Notation**

If S is a set, the notation $x \in S$ means that x is an element of S. The notation $x \notin S$ means that x is not an element of S. A set may be specified using the **set-roster notation** by writing all of its elements between braces. For example, $\{1, 2, 3\}$ denotes the set whose elements are 1, 2, and 3. A variation of the notation is sometimes used to describe a very large set, as when we write $\{1, 2, 3, \ldots, 100\}$ to refer to the set of all integers from 1 to 100. A similar notation can also describe an infinite set, as when we write $\{1, 2, 3, \ldots\}$ to refer to the set of all positive integers. (The symbol $\ldots$ is called an **ellipsis** and is read "and so forth.")

The **axiom of extension** says that a set is completely determined by what its elements are—not the order in which they might be listed or the fact that some elements might be listed more than once.

Example 1 – *Using the Set-Roster Notation*

a. Let $A = \{1, 2, 3\}$, $B = \{3, 1, 2\}$, and $C = \{1, 1, 2, 3, 3, 3\}$. What are the elements of A, B, and C? How are A, B, and C related?

b. Is $\{0\} = 0$?

c. How many elements are in the set $\{1, \{1\}\}$?

d. For each nonnegative integer n, let $U_n = \{n, -n\}$. Find U_1, U_2, and U_0.

Solution:

a. A, B, and C have exactly the same three elements: 1, 2, and 3. Therefore, A, B, and C are simply different ways to represent the same set.

Example 1 – *Solution*

cont'd

b. $\{0\} \neq 0$ because $\{0\}$ is a set with one element, namely 0, whereas 0 is just the symbol that represents the number zero.

c. The set $\{1, \{1\}\}$ has two elements: 1 and the set whose only element is 1.

d. $U_1 = \{1, -1\}$, $U_2 = \{2, -2\}$, $U_0 = \{0, -0\} = \{0, 0\} = \{0\}$.

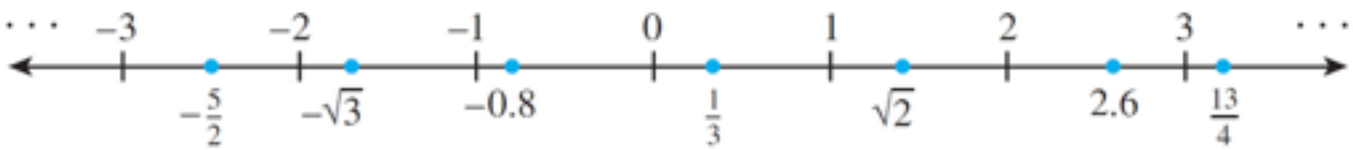
The Language of Sets

Certain sets of numbers are so frequently referred to that they are given special symbolic names. These are summarized in the following table:

Symbol	Set
R	set of all real numbers
Z	set of all integers
Q	set of all rational numbers, or quotients of integers

The Language of Sets

The set of real numbers is usually pictured as the set of all points on a line, as shown below.



The number 0 corresponds to a middle point, called the *origin*.

A unit of distance is marked off, and each point to the right of the origin corresponds to a positive real number found by computing its distance from the origin.

The Language of Sets

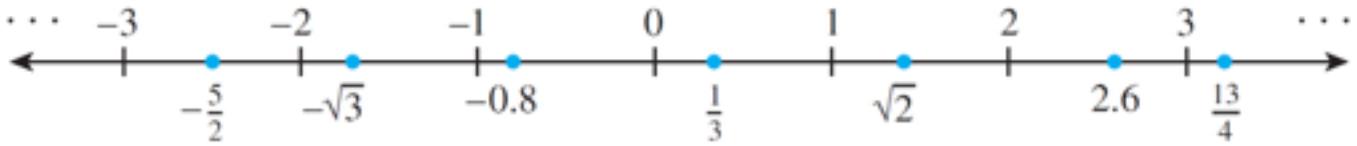
Each point to the left of the origin corresponds to a negative real number, which is denoted by computing its distance from the origin and putting a minus sign in front of the resulting number.

The set of real numbers is therefore divided into three parts: the set of positive real numbers, the set of negative real numbers, and the number 0.

Note that 0 is neither positive nor negative.

The Language of Sets

Labels are given for a few real numbers corresponding to points on the line shown below.



The real number line is called *continuous* because it is imagined to have no holes.

The set of integers corresponds to a collection of points located at fixed intervals along the real number line.

The Language of Sets

Thus every integer is a real number, and because the integers are all separated from each other, the set of integers is called *discrete*. The name *discrete mathematics* comes from the distinction between continuous and discrete mathematical objects.

Another way to specify a set uses what is called the *set-builder notation*.

• Set-Builder Notation

Let S denote a set and let $P(x)$ be a property that elements of S may or may not satisfy. We may define a new set to be **the set of all elements x in S such that $P(x)$ is true**. We denote this set as follows:

$$\{x \in S \mid P(x)\}$$

the set of all — such that

Example 2 – *Using the Set-Builder Notation*

Given that **R** denotes the set of all real numbers, **Z** the set of all integers, and $\mathbf{Z}^+$ the set of all positive integers, describe each of the following sets.

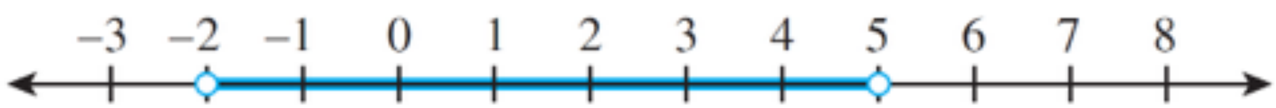
a. $\{x \in \mathbf{R} \mid -2 < x < 5\}$

b. $\{x \in \mathbf{Z} \mid -2 < x < 5\}$

c. $\{x \in \mathbf{Z}^+ \mid -2 < x < 5\}$

Example 2 – *Solution*

a. $\{x \in \mathbf{R} \mid -2 < x < 5\}$ is the open interval of real numbers (strictly) between –2 and 5. It is pictured as follows:



b. $\{x \in \mathbf{Z} \mid -2 < x < 5\}$ is the set of all integers (strictly) between –2 and 5. It is equal to the set

{–1, 0, 1, 2, 3, 4}.

c. Since all the integers in **Z**$^+$ are positive,

$$\{x \in \mathbf{Z}^+ \mid -2 < x < 5\} = \{1, 2, 3, 4\}.$$

Subsets

A basic relation between sets is that of subset.

• Definition

If A and B are sets, then A is called a **subset** of B, written $\boldsymbol{A \subseteq B}$, if, and only if, every element of A is also an element of B.

Symbolically:

$$A \subseteq B \quad \text{means that} \quad \text{For all elements } x, \text{ if } x \in A \text{ then } x \in B.$$

The phrases A *is contained in* B and B *contains* A are alternative ways of saying that A is a subset of B.

Subsets

It follows from the definition of subset that for a set *A* not to be a subset of a set *B* means that there is at least one element of *A* that is not an element of *B*.

Symbolically:

$A \nsubseteq B$ means that There is at least one element x such that $x \in A$ and $x \notin B$.

• Definition

Let A and B be sets. A is a **proper subset** of B if, and only if, every element of A is in B but there is at least one element of B that is not in A.

Example 4 – *Distinction between $\in$ and $\subseteq$*

Which of the following are true statements?

a. $2 \in \{1, 2, 3\}$ **b.** $\{2\} \in \{1, 2, 3\}$ **c.** $2 \subseteq \{1, 2, 3\}$

d. $\{2\} \subseteq \{1, 2, 3\}$ **e.** $\{2\} \subseteq \{\{1\}, \{2\}\}$ **f.** $\{2\} \in \{\{1\}, \{2\}\}$

Solution:

Only (**a**), (**d**), and (**f**) are true.

For (**b**) to be true, the set $\{1, 2, 3\}$ would have to contain the element $\{2\}$. But the only elements of $\{1, 2, 3\}$ are 1, 2, and 3, and 2 is not equal to $\{2\}$. Hence (**b**) is false.

Example 4 – *Solution*

cont'd

For (**c**) to be true, the number 2 would have to be a set and every element in the set 2 would have to be an element of {1, 2, 3}. This is not the case, so (**c**) is false.

For (**e**) to be true, every element in the set containing only the number 2 would have to be an element of the set whose elements are {1} and {2}. But 2 is not equal to either {1} or {2}, and so (**e**) is false.

Cartesian Products

• Notation

Given elements a and b, the symbol (a, b) denotes the **ordered pair** consisting of a and b together with the specification that a is the first element of the pair and b is the second element. Two ordered pairs (a, b) and (c, d) are equal if, and only if, $a = c$ and $b = d$. Symbolically:

$$(a, b) = (c, d) \quad \text{means that} \quad a = c \text{ and } b = d.$$

Example 5 – *Ordered Pairs*

a. Is (1, 2) = (2, 1)?

b. Is $\left(3, \frac{5}{10}\right) = \left(\sqrt{9}, \frac{1}{2}\right)$?

c. What is the first element of (1, 1)?

Solution:

a. No. By definition of equality of ordered pairs,

(1, 2) = (2,1) if, and only if, 1 = 2 and 2 = 1.

But 1 $\neq$ 2, and so the ordered pairs are not equal.

Example 5 – *Solution*

cont'd

b. Yes. By definition of equality of ordered pairs,

$\left(3, \frac{5}{10}\right) = \left(\sqrt{9}, \frac{1}{2}\right)$ if, and only if, $3 = \sqrt{9}$ and $\frac{5}{10} = \frac{1}{2}$.

Because these equations are both true, the ordered pairs are equal.

c. In the ordered pair (1, 1), the first and the second elements are both 1.

Cartesian Products

• Definition

Given sets A and B, the **Cartesian product of $\boldsymbol{A}$ and $\boldsymbol{B}$**, denoted $\boldsymbol{A \times B}$ and read "A cross B," is the set of all ordered pairs (a, b), where a is in A and b is in B. Symbolically:

$$\boldsymbol{A \times B} = \{(a, b) \mid a \in A \text{ and } b \in B\}.$$

Example 6 – *Cartesian Products*

Let $A = \{1, 2, 3\}$ and $B = \{u, v\}$.

a. Find $A \times B$

b. Find $B \times A$

c. Find $B \times B$

d. How many elements are in $A \times B$, $B \times A$, and $B \times B$?

e. Let **R** denote the set of all real numbers. Describe $\mathbf{R} \times \mathbf{R}$.

Example 6 – *Solution*

a. $A \times B = \{(1, u), (2, u), (3, u), (1, v), (2, v), (3, v)\}$

b. $B \times A = \{(u, 1), (u, 2), (u, 3), (v, 1), (v, 2), (v, 3)\}$

c. $B \times B = \{(u, u), (u, v), (v, u), (v, v)\}$

d. $A \times B$ has six elements. Note that this is the number of elements in A times the number of elements in B.

$B \times A$ has six elements, the number of elements in B times the number of elements in A. $B \times B$ has four elements, the number of elements in B times the number of elements in B.

Example 6 – *Solution*

cont'd

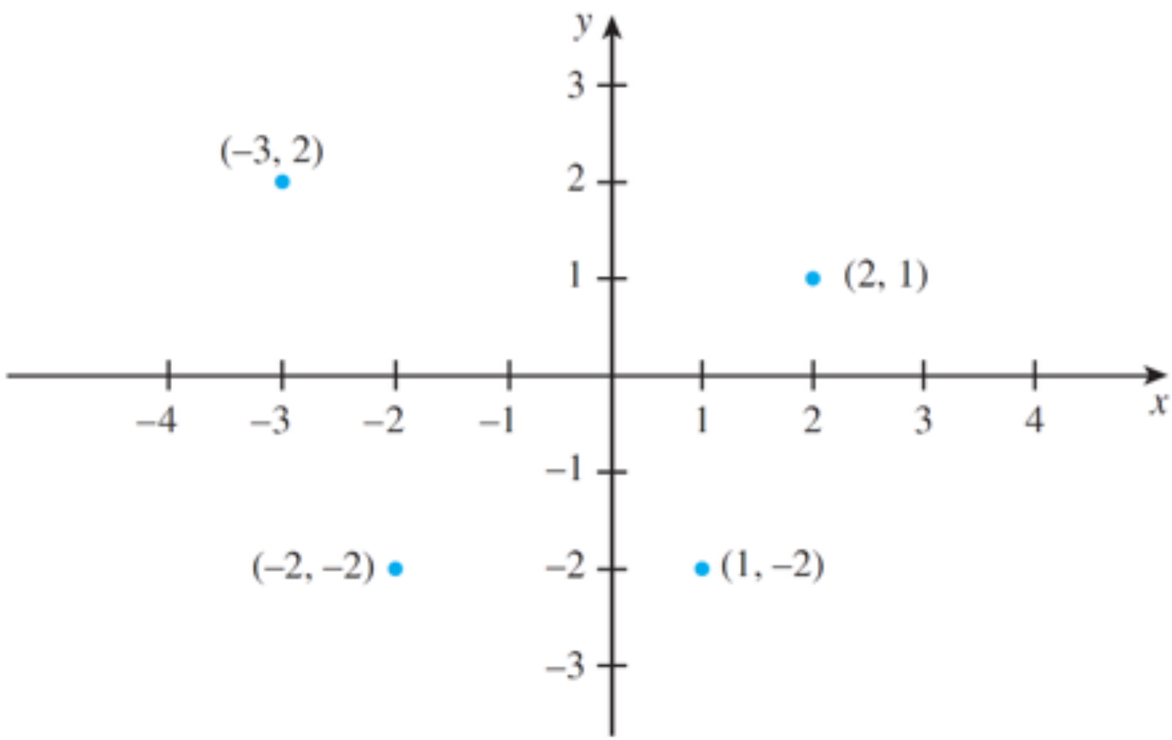
e. **R** × **R** is the set of all ordered pairs (x, y) where both x and y are real numbers.

If horizontal and vertical axes are drawn on a plane and a unit length is marked off, then each ordered pair in **R** × **R** corresponds to a unique point in the plane, with the first and second elements of the pair indicating, respectively, the horizontal and vertical positions of the point.

Example 6 – *Solution*

cont'd

The term **Cartesian plane** is often used to refer to a plane with this coordinate system, as illustrated in Figure 1.2.1.



A Cartesian Plane

Figure 1.2.1

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The Language of Relations and Functions

The objects of mathematics may be related in various ways.

A set A may be said to be related to a set B if A is a subset of B, or if A is not a subset of B, or if A and B have at least one element in common.

A number x may be said to be related to a number y if $x < y$, or if x is a factor of y, or if $x^2 + y^2 = 1$.

Let $A = \{0, 1, 2\}$ and $B = \{1, 2, 3\}$ and let us say that an element x in A is related to an element y in B if, and only if, x is less than y.

The Language of Relations and Functions

Let us use the notation *x R y* as a shorthand for the sentence "*x* is related to *y*." Then

$$\begin{array}{llll} 0\ R\ 1 & \text{since} & 0 < 1, & \\ 0\ R\ 2 & \text{since} & 0 < 2, & \\ 0\ R\ 3 & \text{since} & 0 < 3, & \\ 1\ R\ 2 & \text{since} & 1 < 2, & \\ 1\ R\ 3 & \text{since} & 1 < 3, & \text{and} \\ 2\ R\ 3 & \text{since} & 2 < 3. & \end{array}$$

On the other hand, if the notation $x \not R\ y$ represents the sentence "*x* is not related to *y*," then

$$\begin{array}{llll} 1 \not R\ 1 & \text{since} & 1 \not< 1, & \\ 2 \not R\ 1 & \text{since} & 2 \not< 1, & \text{and} \\ 2 \not R\ 2 & \text{since} & 2 \not< 2. & \end{array}$$

The Language of Relations and Functions

The Cartesian product of *A* and *B*, $A \times B$, consists of all ordered pairs whose first element is in *A* and whose second element is in *B*:

$$A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}.$$

In this case,

$$A \times B = \{(0, 1), (0, 2), (0, 3), (1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)\}.$$

The elements of some ordered pairs in $A \times B$ are related, whereas the elements of other ordered pairs are not. Consider the set of all ordered pairs in $A \times B$ whose elements are related

$$\{(0, 1), (0, 2), (0, 3), (1, 2), (1, 3), (2, 3)\}.$$

The Language of Relations and Functions

Observe that knowing which ordered pairs lie in this set is equivalent to knowing which elements are related to which.

The relation itself can therefore be thought of as the totality of ordered pairs whose elements are related by the given condition.

- **Definition**

Let A and B be sets. A **relation R from A to B** is a subset of $A \times B$. Given an ordered pair (x, y) in $A \times B$, **x is related to y by R**, written $x \; R \; y$, if, and only if, (x, y) is in R. The set A is called the domain of R and the set B is called its co-domain.

The Language of Relations and Functions

The notation for a relation R may be written symbolically as follows:

$$x\ R\ y \text{ means that } (x, y) \in R.$$

The notation $x \not R\, y$ means that x is not related to y by R:

$$x \not R\, y \text{ means that } (x, y) \notin R.$$

Example 1 – *A Relation as a Subset*

Let $A = \{1, 2\}$ and $B = \{1, 2, 3\}$ and define a relation R from A to B as follows: Given any $(x, y) \in A \times B$,

$$(x, y) \in R \quad \text{means that} \quad \frac{x - y}{2} \text{ is an integer.}$$

a. State explicitly which ordered pairs are in $A \times B$ and which are in R.

b. Is $1\ R\ 3$? Is $2\ R\ 3$? Is $2\ R\ 2$?

c. What are the domain and co-domain of R?

Example 1 – *Solution*

a. $A \times B = \{(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3)\}$. To determine explicitly the composition of R, examine each ordered pair in $A \times B$ to see whether its elements satisfy the defining condition for R.

$$(1, 1) \in R \text{ because } \tfrac{1-1}{2} = \tfrac{0}{2} = 0, \text{ which is an integer.}$$

$$(1, 2) \notin R \text{ because } \tfrac{1-2}{2} = \tfrac{-1}{2}, \text{ which is not an integer.}$$

$$(1, 3) \in R \text{ because } \tfrac{1-3}{2} = \tfrac{-2}{2} = -1, \text{ which is an integer.}$$

$$(2, 1) \notin R \text{ because } \tfrac{2-1}{2} = \tfrac{1}{2}, \text{ which is not an integer.}$$

$$(2, 2) \in R \text{ because } \tfrac{2-2}{2} = \tfrac{0}{2} = 0, \text{ which is an integer.}$$

Example 1 – *Solution*

cont'd

$$(2, 3) \notin R \text{ because } \frac{2-3}{2} = \frac{-1}{2}, \text{ which is not an integer.}$$

Thus

$$R = \{(1, 1), (1, 3), (2, 2)\}$$

b. Yes, $1 \, R \, 3$ because $(1, 3) \in R$.

No, $2 \not R \, 3$ because $(2, 3) \notin R$.

Yes, $2 \, R \, 2$ because $(2, 2) \in R$.

c. The domain of R is $\{1, 2\}$ and the co-domain is $\{1, 2, 3\}$.

Arrow Diagram of a Relation

Arrow Diagram of a Relation

Suppose R is a relation from a set A to a set B. The **arrow diagram for *R*** is obtained as follows:

1. Represent the elements of A as points in one region and the elements of B as points in another region.

2. For each x in A and y in B, draw an arrow from x to y if, and only if, x is related to y by R. Symbolically:

Draw an arrow from x to y
if, and only if, $x \, R \, y$
if, and only if, $(x, y) \in R$.

Example 3 – *Arrow Diagrams of Relations*

Let $A = \{1, 2, 3\}$ and $B = \{1, 3, 5\}$ and define relations S and T from A to B as follows:

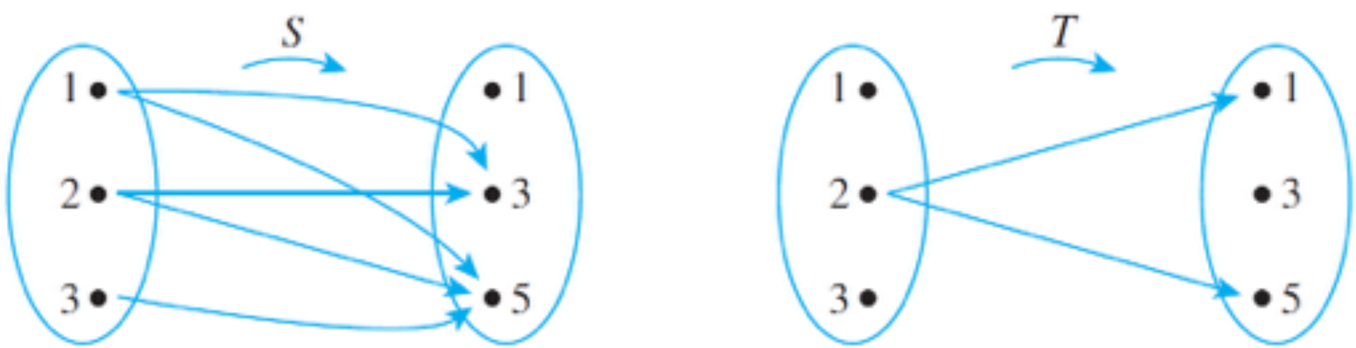
For all $(x, y) \in A \times B$,

$(x, y) \in S$ means that $x < y$ S is a "less than" relation.

$T = \{(2, 1), (2, 5)\}$.

Draw arrow diagrams for S and T.

Example 3 – *Solution*



These example relations illustrate that it is possible to have several arrows coming out of the same element of *A* pointing in different directions.

Also, it is quite possible to have an element of *A* that does not have an arrow coming out of it.

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Functions

- **Definition**

A **function F from a set A to a set B** is a relation with domain A and co-domain B that satisfies the following two properties:

1. For every element x in A, there is an element y in B such that $(x, y) \in F$.
2. For all elements x in A and y and z in B,

$$\text{if} \quad (x, y) \in F \text{ and } (x, z) \in F, \quad \text{then} \quad y = z.$$

Functions

Properties (1) and (2) can be stated less formally as follows: A relation *F* from *A* to *B* is a function if, and only if:

1. Every element of *A* is the first element of an ordered pair of *F*.

2. No two distinct ordered pairs in *F* have the same first element.

• Notation

If A and B are sets and F is a function from A to B, then given any element x in A, the unique element in B that is related to x by F is denoted $\boldsymbol{F(x)}$, which is read "***F* of *x***."

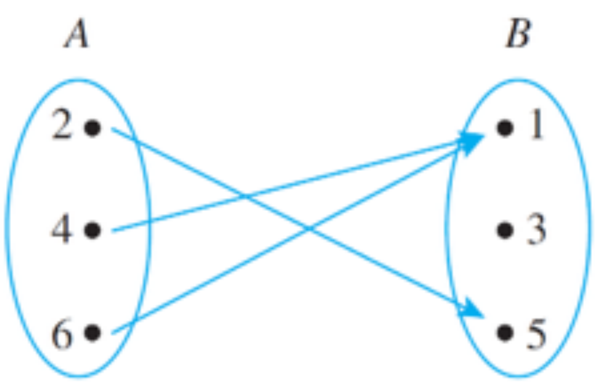
Example 4 – *Functions and Relations on Finite Sets*

Let $A = \{2, 4, 6\}$ and $B = \{1, 3, 5\}$. Which of the relations R, S, and T defined below are functions from A to B?

a. $R = \{(2, 5), (4, 1), (4, 3), (6, 5)\}$

b. For all $(x, y) \in A \times B$, $(x, y) \in S$ means that $y = x + 1$.

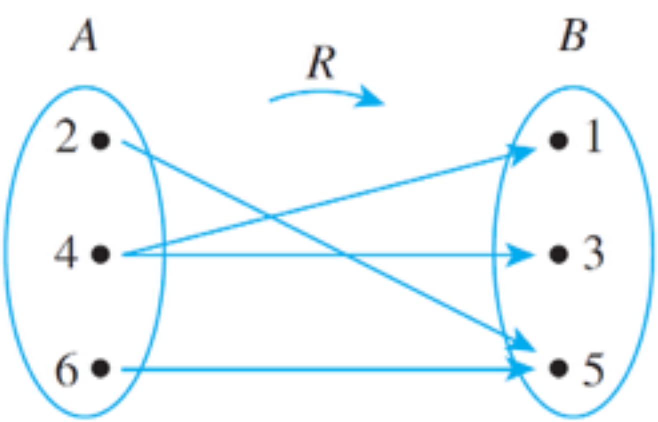
c. T is defined by the arrow diagram



Example 4(a) – *Solution*

R is not a function because it does not satisfy property (2). The ordered pairs (4, 1) and (4, 3) have the same first element but different second elements.

You can see this graphically if you draw the arrow diagram for *R*. There are two arrows coming out of 4: One points to 1 and the other points to 3.

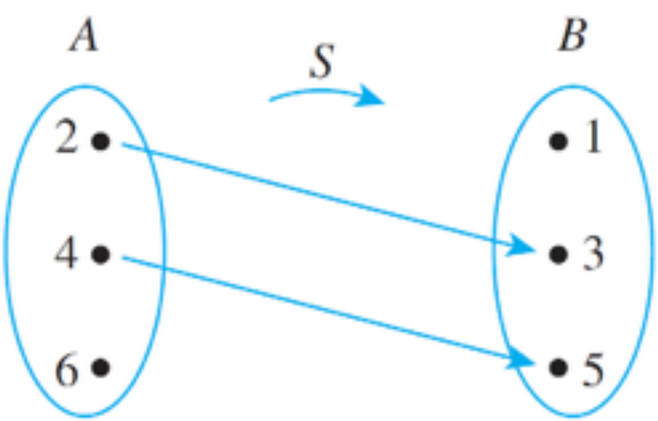


Example 4(b) – *Solution*

cont'd

S is not a function because it does not satisfy property (1). It is not true that every element of A is the first element of an ordered pair in S.

For example, $6 \in A$ but there is no y in B such that $y = 6 + 1 = 7$. You can also see this graphically by drawing the arrow diagram for S.



Example 4(c) – *Solution*

cont'd

T is a function: Each element in {2, 4, 6} is related to some element in {1, 3, 5} and no element in {2, 4, 6} is related to more than one element in {1, 3, 5}.

When these properties are stated in terms of the arrow diagram, they become (1) there is an arrow coming out of each element of the domain, and (2) no element of the domain has more than one arrow coming out of it.

So you can write $T(2) = 5$, $T(4) = 1$, and $T(6) = 1$.

Function Machines

Function Machines

Another useful way to think of a function is as a machine. Suppose f is a function from X to Y and an input x of X is given.

Imagine f to be a machine that processes x in a certain way to produce the output $f(x)$. This is illustrated in Figure 1.3.1

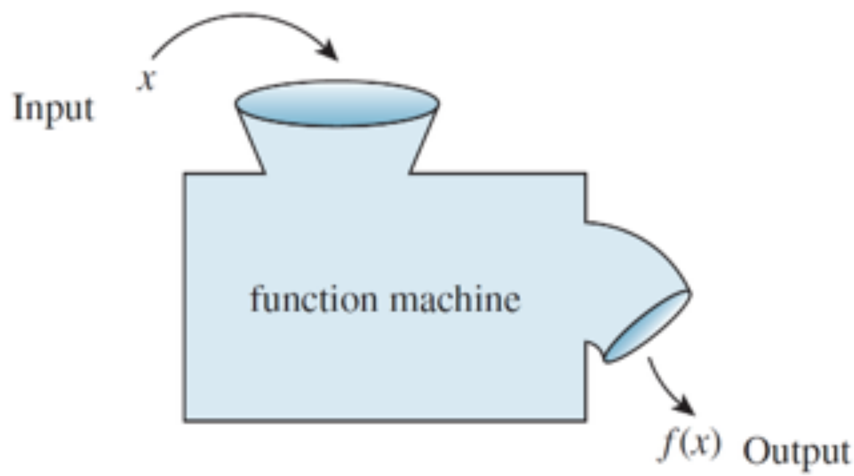


Figure 1.3.1

Example 6 – *Functions Defined by Formulas*

The **squaring function** f from **R** to **R** is defined by the formula $f(x) = x^2$ for all real numbers x.

This means that no matter what real number input is substituted for x, the output of f will be the square of that number.

This idea can be represented by writing $f(\bullet\ ) = \bullet\ ^2$. In other words, f sends each real number x to x^2, or, symbolically, $f : x \rightarrow x^2$. Note that the variable x is a dummy variable; any other symbol could replace it, as long as the replacement is made everywhere the x appears.

Example 6 – *Functions Defined by Formulas* cont'd

The **successor function** g from **Z** to **Z** is defined by the formula $g(n) = n + 1$. Thus, no matter what integer is substituted for n, the output of g will be that number plus one: $g(\bullet\ ) = \bullet + 1$.

In other words, g sends each integer n to $n + 1$, or, symbolically, $g : n \to n + 1$.

An example of a **constant function** is the function h from **Q** to **Z** defined by the formula $h(r) = 2$ for all rational numbers r.

Example 6 – *Functions Defined by Formulas* cont'd

This function sends each rational number *r* to 2. In other words, no matter what the input, the output is always 2: *h*(•) = 2 or h : *r* → 2.

The functions *f*, *g*, and *h* are represented by the function machines in Figure 1.3.2.

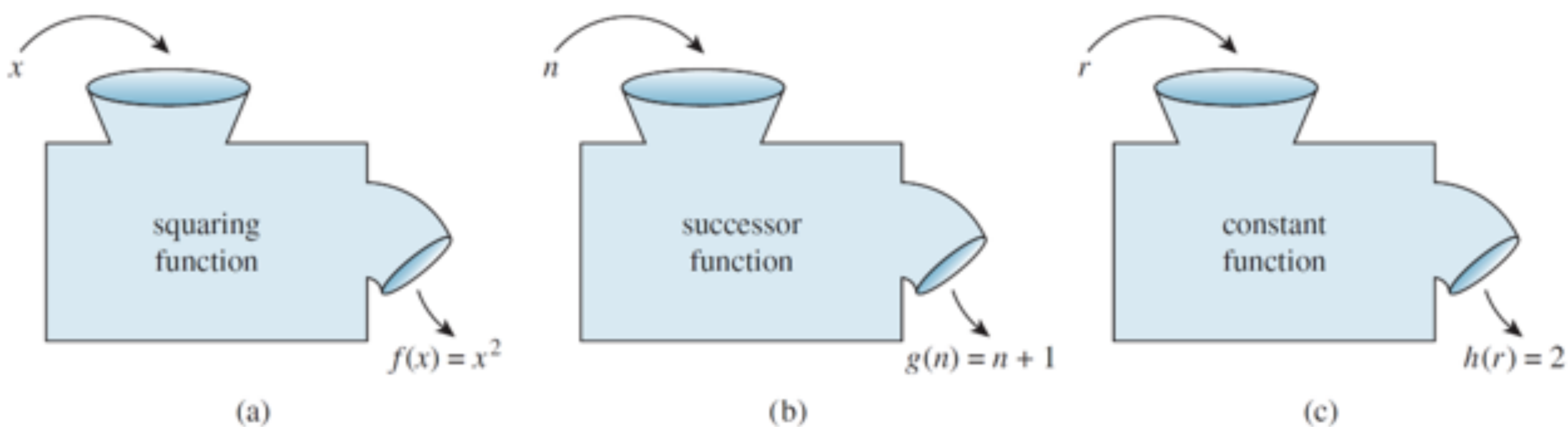


Figure 1.3.2

Function Machines

A function is an entity in its own right. It can be thought of as a certain relationship between sets or as an input/output machine that operates according to a certain rule.

This is the reason why a function is generally denoted by a single symbol or string of symbols, such as f, G, of log, or sin.

A relation is a subset of a Cartesian product and a function is a special kind of relation.

Function Machines

Specifically, if f and g are functions from a set A to a set B, then

$$f = \{(x, y) \in A \times B \mid y = f(x)\}$$

and

$$g = \{(x, y) \in A \times B \mid y = g(x)\}.$$

It follows that

f **equals** g,   written $f = g$,
if, and only if, $f(x) = g(x)$ for all x in A.

Example 7 – *Equality of Functions*

Define $f : \mathbf{R} \to \mathbf{R}$ and $g: \mathbf{R} \to \mathbf{R}$ by the following formulas:

$$f(x) = |x| \quad \text{for all } x \in \mathbf{R}.$$
$$g(x) = \sqrt{x^2} \quad \text{for all } x \in \mathbf{R}.$$

Does $f = g$?

Solution:

Yes. Because the absolute value of any real number equals the square root of its square,

$|x| = \sqrt{x^2}$ for all $x \in \mathbf{R}$. Hence $f = g$.